J. Phys. A: Math. Theor. **41** (2008) 235205 (13pp) doi:10.1088/1751-8113/41/23/235205

Linear frictional forces cause orbits to neither circularize nor precess

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Received 25 March 2008, in final form 21 April 2008
Published 15 May 2008
Online at stacks.iop.org/JPhysA/41/235205

Abstract
For the undamped Kepler potential the lack of precession has historically been understood in terms of the Runge–Lenz symmetry. For the damped Kepler problem this result may be understood in terms of the generalization of Poisson structure to damped systems suggested recently by Tarasov (2005 *J. Phys. A: Math. Gen.* **38** 2145). In this generalized algebraic structure the orbit-averaged Runge–Lenz vector remains a constant in the linearly damped Kepler problem to leading order in the damping coefficient. Beyond Kepler, we prove that, for any potential proportional to a power of the radius, the orbit shape and precession angle remain constant to leading order in the linear friction coefficient.

PACS numbers: 46.40.Ff, 45.20.Jj, 45.50.Pk, 34.60.+z

1. Introduction

What happens to orbits subject to linear frictional drag? In typical physical settings, such as Rydberg atoms or stellar binaries, the effective frictional forces are nonlinear and, typically, lead to the circularization of the orbit. Orbital evolution under linear friction is special in that, as we show below, the eccentricity and the apsides do not change to leading order in the damping. The purpose of this paper is to understand this elementary result from the underlying dynamical symmetry of the Kepler problem, thus demonstrating the utility of a Hamiltonian notion in its non-Hamiltonian generalization.

In many astrophysical situations, the secular evolution due to friction of orbits in a two-body system is towards circular orbits. In an orbit in a central field the angular momentum scales with the momentum while the energy generally scales with the momentum squared. Friction, assumed to be spatially isotropic and homogeneous but time odd, typically scales the momentum. This means generally that the resultant secular evolution in central force systems is that in which the energy is minimized at a fixed angular momentum. This is clearly the

1751-8113/08/235205+13$30.00

circular orbit. The velocity dependence of the frictional force is quite relevant, in particular as referenced against the velocity dispersion of the (undamped) motion in that central potential. Clearly, under the action of such dissipative forces, a consequence of symmetry is that the flow in orbital shape (not size) has two fixed points, circular orbits and strictly radial (infall) orbits.

Few physical problems have received more scrutiny than bounded orbits in the two- and few-body system. Among these, the two-body Kepler problem is arguably the most experimentally relevant and best studied example, having been illuminated by intense theoretical inquiry spanning hundreds of years leading to important insights even in relatively recent times [2–22]. We do not present a systematic review or histiography of this celebrated problem (though we thankfully acknowledge also [23–30] which we have found quite useful for our study). We do not aim to contribute to the vast literature on astrophysically and microphysically relevant models of friction in orbital problems (though the interested reader may find [31–43] a useful launching point for such a review).

Instead, our purpose here is to accommodate from the dynamical symmetry group point-of-view the result that linear frictional damping (to leading order) preserves the orbit's shape. Although Hamiltonian systems may lose dynamical symmetry completely when dissipative forces are included, it can be shown that some structure may remain under a modified symplectic form. After a brief introduction to the method by which Tarasov extends symplectic structure of Hamiltonian mechanics to dissipative systems, we apply it to the determination of the time averages of dynamical quantities. Damping invariably introduces new dynamical timescales and the time averaging we implement is over times short compared with these timescales (but still long compared with the orbital timescales in the undamped problem). Tarasov's construction reveals the relevance of the dynamical symmetry algebra to the damped Kepler problem.

We then compare this approach to the classic 'variations of constants' method of orbit parameter evolution by describing an improvement that follows from our study. The elementary method can be generalized to non-Kepler homogeneous potentials and also determines orbital shape evolution for linearly damped Kepler orbits beyond leading order.

2. Dynamical symmetry and Tarasov's construction

In the undamped Kepler problem the lack of precession is generally understood as a consequence of a dynamical symmetry, the celebrated $so(4)$ symmetry formed from the two commuting $so(3)$, one from the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ the other from the Runge–Lenz vector, $\vec{S} = \vec{L} \times \vec{p} + k\frac{\vec{r}}{|\vec{r}|}$ [6–8] being the maximal set of local, algebraically-independent operators that commute with the Hamiltonian, $H = \frac{\vec{p}^2}{2} + V(r)$, with $V(r) = kr^{\alpha}$ for $k < 0$ and $\alpha = -1$. (Though see [22] for a more precise and general statement of the connection between algebra and orbits in a central field.)

$$\{L_i, L_j\} = 2\epsilon_{ijk}L_k \qquad \{L_i, S_j\} = 2\epsilon_{ijk}S_k \qquad \{S_i, S_j\} = -2H\epsilon_{ijk}S_k. \tag{2.1}$$

The length of $\vec{S}$ is proportional to the eccentricity (and points along the semi-major axis of the orbit in the direction to the periastron from the focus). Defining $\vec{L}$ and $\vec{S}$ has utility beyond their being constants in the 2-body Kepler problem, for example, parameterizing the secular evolution of orbits under various Hamiltonian perturbations [29, 44]. This $so(4)$ is one of the maximal compact factor groups of the $so(4, 2)$ (the conformal group) extended symmetry formed by $\vec{L}$, $\vec{A}$, H, the generalization of the scaling operator $\mathcal{R} = \vec{r} \cdot \vec{p}$ and the virial operator $\mathcal{V} = \frac{\vec{p}^2}{2} - \frac{r}{2}\partial_r V(r)$ [17, 20].

The other central potential possessing an easily recognizable dynamical symmetry is the multi-dimensional harmonic oscillator (V as given with $k > 0$, $\alpha = 2$). As is well known, the isotropic D-dimensional harmonic oscillator's naive $O(D)$ symmetry is part of a larger $U(D)$ dynamical symmetry. For $D = 2$ harmonic oscillator, note that the $U(2)$ symmetry does enlarge further to a $so(3, 2)$ when including $\mathcal{R}, \mathcal{V}$ and their generalization (the virial subalgebra equations (4.16) through (4.19) of each oscillator alone and closes to a $sl(2, R)$ subgroup of the $so(3, 2)$). Note further that it is this later algebra that is isomorphic to the dimensionally reduced $so(4, 2)$ of the 3D Kepler problem, by which we mean the reduction of that algebra to generators associated with the orbital plane only. These considerations can also be understood from the KS construction [26, 27] of the Kepler problem, in which a four-dimensional isotropic harmonic oscillator is the starting point. In that construction the $u(4) = su(4)\times u(1)$ is, of itself, not preserved by the KS construction. Instead, it is the $u(2, 2)$ subgroup of the four identical, independent oscillator's $sp(8, \mathrm{R})$ symmetry in which the overall $u(1)$ can be isolated as the angular momentum constraint of the KS construction [28]. The residual symmetry $su(2, 2) \sim so(4, 2)$ is that of the 3D Kepler problem. The analytical connection between the Kepler problem and the isotropic harmonic oscillator has deep historical roots, going back to Newton and Hooke (see [45] and references therein). Finally, the geometric construction of the undamped Kepler problem as geodesic flow on (spatial) a 3-manifold of constant curvature relates the $so(4)$ dynamical symmetry to the isometry group generated by Killing vector fields on the spatial slice [19, 21, 23]. These various connections between the Kepler problem and the isotropic harmonic oscillator do not lead to a simple structural connection between the associated damped problems.

To leading order in the damping, Kepler orbits subject to linear frictional force do not change shape or precess as they decay. It would be satisfying to understand this elementary result as a consequence of the preservation of the dynamical algebra under linear friction. Although this is reminiscent of the damped N-dimensional harmonic oscillator, there is no simple way to relate the damped problems. Since the subgroup associated with the shape and precession (through the $\vec{S}$) is rank one it is suggestive that the entire group structure is preserved to leading order in the linear friction.

A recent paper by Tarasov [1] suggests a straightforward generalization of the Poisson structure to systems with dissipative forces. There are many other approaches to addressing structural questions of dissipative systems (for one example, see [46, 47]). We find the approach of [1] to be most useful for addressing questions of the dynamical symmetries that survive including dissipation. For completeness we now briefly review Tarasov's construction, and apply it to dissipation in the central field problem in the following section.

To preserve as much of the algebraic structure as possible, Tarasov constructs a one-parameter family of two forms (that define a generalized Poisson structure) that, in a sense, interpolate between different dampings. In the zero damping limit it smoothly matches onto the canonical symplectic form. Dimensionally, any damping parameter introduces a new timescale into the problem, thus this new interpolating 2-form must also be explicitly time dependent. Tarasov requires this family of 2-forms to have the following useful properties:

(1) Non-degeneracy: the 2-form $\omega = \omega_{ij}(t)\, \mathrm{d}x^i \wedge \mathrm{d}x^j$ is antisymmetric and non-degenerate along the entire flow. The x^i are the $2N$ (local) phase-space coordinates. In positive terms, the inverse $\omega^{ij}\omega_{jk} = \delta^i_j$ exists almost globally[3].

[3] Since we do not formulate this entirely in the exterior calculus, we must allow for higher codimension singularities that may not be resolvable in the dissipative system.

(2) Jacobi Identity: the 2-form is used to define a new Poisson bracket $\{A, B\}_T = \omega^{ij}\partial_i A \partial_j B$ that forms an associative algebra. Explicitly it satisfies

$$\{A, \{B, C\}_T\}_T + \{B, \{C, A\}_T\}_T + \{C, \{A, B\}_T\}_T = 0. \tag{2.2}$$

Here we use the subscript 'T' to distinguish this bracket from the Poisson bracket of the undamped problem.

(3) Derivation property of time translation: with respect to this new bracket the time derivative of the new Poisson bracket satisfies the derivation property (also called the Liebnitz rule)

$$\frac{\mathrm{d}}{\mathrm{d}t}\{A, B\}_T = \left\{\frac{\mathrm{d}A}{\mathrm{d}t}, B\right\}_T + \left\{A, \frac{\mathrm{d}B}{\mathrm{d}t}\right\}_T. \tag{2.3}$$

These requirements are remarkable for several reasons. First, property (1) indicates that (2) and (3) are possible. The deeper relevance of property (1) is that we can regard the 2-form as (essentially a) global metric on the phase space. Property (2) indicates local mechanical observables in this 'dissipation deformed' algebra form a Lie algebra. Property (3) is a key to the utility of Tarasov's construction for understanding constants of motion in dissipative systems. It stipulates that time development in the dissipative system, while no longer just $\{, H\}$ (or even $\{, H\}_T$), must be compatible with the structure of the symplectic algebra in the new bracket and thus the (new) brackets of time-independent quantities in the dissipative system are themselves time independent. Thus, just as in the Hamiltonian case, time-independent quantities form a closed subalgebra. Note that for a Hamiltonian system property (3) is automatic since in that case time translation is an inner automorphism of the symplectic algebra. In a dissipative system by contrast the Hamiltonian is no longer the operator of time translation, but, if Tarasov's construction can be implemented, time translation is still an automorphism of the algebra, and as such may be regarded as an outer automorphism. Finally, from property (3) it follows after a brief calculation that the 2-form ω must be time independent in the full dissipative system, $\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$. In terms of symplectic geometry, this is metric compatibility of the dissipative flow.

To proceed with the construction, consider the general flow $\dot{x}^i = \chi^i(\vec{x}, t)$. Again, these are not assumed to be Hamiltonian flows. Assuming property (1) and using ω^{ij} to form a bracket $\{A, B\}_T = \omega^{ij}\partial_i A \partial_j B$, property (2) leads to the condition

$$\omega^{im}\partial_m\omega^{jk} + \omega^{jm}\partial_m\omega^{ki} + \omega^{km}\partial_m\omega^{ij} = 0. \tag{2.4}$$

Total time derivatives and derivatives along phase-space directions do not commute in the flow,

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}, \partial_i\right] A = -\partial_j A \partial_i \chi^j. \tag{2.5}$$

Using this and the Jacobi Identity (2.2), one sees that property (3) implies a condition relating the form ω and the flow χ^i,

$$\frac{\partial\omega_{ij}(t)}{\partial t} = \partial_i\chi_j - \partial_j\chi_i \qquad \text{where} \quad \chi_j = \omega_{jk}(t)\chi^k. \tag{2.6}$$

Given χ^i, we proceed by solving (2.4) for an ω^{ij} that satisfies (2.6). This completes Tarasov's construction.

We briefly offer a few further remarks helpful to orient the reader. First, in the more familiar context of Hamiltonian flows, there $\dot{x}^i = \chi^i = \{x^i, H\}$ for a local function H on the phase space. For this case we can compute in the Darboux frame and learn that the usual symplectic form (automatically satisfying (2.4)) is a solution also to (2.6) since the RHS in that case is zero. We recognize the RHS of (2.6) as exactly the obstruction to the flow, χ^i being Hamiltonian.

Conformal transformation of the 2-form, $\tilde{\omega} = \Omega\omega$, where Ω is a scalar function, can only relate two solutions of (2.4) and (2.6) if and only if Ω is a constant of the motion $\frac{\mathrm{d}\Omega}{\mathrm{d}t} = 0$. For in that case (2.6) indicates that

$$\frac{\partial\Omega}{\partial t}\omega_{ij} = \chi_j\partial_i\Omega - \chi_i\partial_j\Omega \tag{2.7}$$

whereas (2.4) yields

$$\omega_{jk}\partial_j\Omega + \omega_{kl}\partial_j\Omega + \omega_{lj}\partial_k\Omega = 0 \tag{2.8}$$

so, contracting by χ^k and comparing with (2.7), we learn that Ω must be a constant of the motion. Thus, each solution is conformally unique.

We do not know what conditions on χ^i lead to the existence of even one non-singular *simultaneous* solution ω of (2.4) and (2.6). Tarasov [1] provides an explicit solution for a general Hamiltonian system amended by a general linear frictional force. The general question of the existence of $\omega(t)$ for a more general χ^i is at this point unclear, but beyond the scope of this present effort.

3. Dynamical symmetry in a damped system

Consider damped orbital motion in a central field;

$$\dot{\vec{x}} = \vec{p} \tag{3.1}$$

$$\dot{\vec{p}} = -\partial_r V\frac{\vec{x}}{r} - \beta(p)\vec{p} \tag{3.2}$$

with $r = |\vec{x}|$ and $V(r)$ the interparticle potential (throughout we take the reduced mass to be normalized to 1). The function $\beta(p)$ is some general function parameterizing the speed dependence of the damping, and this form of the damping function is the most general consistent with isotropy and homogeniety of the damping forces. Note that we can understand this set as descending from a limit in which the central mass is very much larger than the orbital mass though, as in general, damping does inextricably mix the center-of-mass motion and the relative motion. We call linear damping the choice of β constant.

Equation (2.6) takes the form

$$\frac{\partial\omega_{xp}(t)}{\partial t} = \partial_x(\omega_{px'}\chi^{x'}) - \partial_p(\omega_{xp'}\chi^{p'}) = \partial_p\omega_{xp'}(\beta(p)p'). \tag{3.3}$$

Again, we do not know if solutions to equation (3.3) exist and satisfy Jacobi for every choice of $\beta(p)$. However, for $\beta(p) = \text{const}$ there is a simple solution to equation (3.3) that satisfies Jacobi [1],

$$\omega_{ij}(t) = \mathrm{e}^{\beta t}\hat{\omega}_{ij} \tag{3.4}$$

where $\hat{\omega}$ is the usual symplectic form of the undamped Kepler problem. Physically this corresponds to the uniform shrinkage of phase-space volumes under linear damping.

Clearly, in going from $\{\,,\}$ (Poisson bracket) to the new bracket $\{\,,\}_T$ the relations in equation (2.1) gain a factor of $\mathrm{e}^{-\beta t}$. The algebra in the new bracket resulting from this simple rescaling is still $so(4)$. The utility of this simple change to the algebra of equation (2.1) (which was for the undamped system) is that it is now compatible with the evolution under equations (3.1) and (3.2) of the damped system. To see this in an example, take the first relation in equation (2.1) and take the (total) time derivative of both sides. Then note $\frac{\mathrm{d}\{L_i,L_j\}}{\mathrm{d}t} = -2\beta(2\epsilon_{ijk}L_k) \neq 2\epsilon_{ijk}\dot{L}_k$; i.e. the usual Poisson bracket is no longer compatible with time evolution. Duplicating the previous line for $\{L_i, L_j\}_T = 2\,\mathrm{e}^{-\beta t}\epsilon_{ijk}L_k$ one learns that this is compatible with the flow equations (3.1) and (3.2). Similarly, one may check that all the brackets in equation (2.1) (after replacing $\{\,,\}$ with $\{\,,\}_T$) are as well. Also note that $\{L_i, H\}_T = 0 = \{S_i, H\}_T$, though since brackets with H no longer delineate time evolution, these equations do not imply that $\vec{L}$ and $\vec{S}$ are constants of the motion in the dissipative system (also clear from equation (3.10)).

The critique here is familiar to any attempt to reconcile symplectic structure and dissipation; fundamentally, equations (3.1) and (3.2) still treat x and p differently so that time evolution is no longer an element in the dynamical algebra of $\{\,,\}$ or $\{\,,\}_T$.

To relax the category of 'constants of the motion' sufficiently for dissipative systems, consider to what extent dynamical quantities averaged over some number of orbits change on a longer timescale, i.e. on a timescale relevant to the dissipation (note $1/\beta(p)$ is essentially that timescale). Let $\langle\,\rangle$ denote time averages over many orbits, $\mathcal{O}$ a classical observable, and suppose that ω is a solution to equation (2.6) and the Jacobi identity for the system as in equations (3.1) and (3.2). In general,

$$\langle\{\mathcal{O}, H\}_T\rangle = \langle\omega^{xp}(t)(\dot{x}\partial_x\mathcal{O} - (-\dot{p} - \beta(p)p)\partial_p\mathcal{O})\rangle \tag{3.5}$$

$$= \left\langle \omega(t)\left[\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}t} - \frac{\partial\mathcal{O}}{\partial t}\right] + \omega^{xp}(t)\beta(p)p\partial_p\mathcal{O}\right\rangle. \tag{3.6}$$

Note that sums are implied in the x, p indices of the $\omega^{xp}(t)$, the new symplectic form. Above we have used isotropy to rewrite the sum in the first term in terms of the (normalized) symplectic trace of $\omega^{xp}(t)$ which we denote simply as $\omega(t)$. To show one intermediate step, integrating by parts and using equation (3.3) we arrive at

$$\langle\{\mathcal{O}, H\}_T\rangle = \frac{1}{T}\Delta(\omega_T\mathcal{O}) + \Big\langle \omega^{x\alpha}\omega^{p\beta}(\partial_\alpha\chi_\beta - \partial_\beta\chi_\alpha + \chi^l\partial_l\omega_{\alpha\beta})\mathcal{O} \\ + \omega^{xp}\beta(p)p\partial_p\mathcal{O} - \omega\frac{\partial\mathcal{O}}{\partial t}\Big\rangle, \tag{3.7}$$

where $\Delta(G)$ refers simply to the overall change of the quantity G over time T. Finally, using equation (2.5) and the fact that ω^{xp} satisfies the Jacobi identity we reduce the above to

$$\langle\{\mathcal{O}, H\}_T\rangle = \frac{1}{T}\Delta(\omega\mathcal{O}) + \left\langle(\omega^{xp'}\partial_{p'}\chi^p - \omega^{px'}\partial_{x'}\chi^x)\mathcal{O} + \omega^{xp}\beta(p)p\partial_p\mathcal{O} - \omega\frac{\partial\mathcal{O}}{\partial t}\right\rangle. \tag{3.8}$$

We now specialize to vector fields of the general form equations (3.1) and (3.2) to find,

$$\frac{1}{T}\Delta(\omega\mathcal{O}) = \left\langle\{\mathcal{O}, H\}_T + \omega\frac{\partial\mathcal{O}}{\partial t} - \omega^{xp}(\partial_p(\beta(p))p\mathcal{O} - \beta(p)p\partial_p\mathcal{O})\right\rangle, \tag{3.9}$$

and so making the RHS zero indicates conserved quantities in the non-Hamiltonian system. Again, this last result was derived for general $\beta(p)$, which assumes only that the friction is isotropic and homogeneous. In the linear friction case $\beta(p) = \beta = \text{const}$. For this case, using $\mathcal{O} = L/\omega^2$ in the above equation implies that L/ω are constants of the motion in this system. Similarly, taking $\mathcal{O} = S/\omega$ indicates that ΔS is proportional to $(2\beta\vec{L}/\omega) \times \langle\omega\vec{p}\rangle$ which, again, is zero to first order in β. This result then applied to the case of bounded Kepler orbits with linear damping indicates that the (orbit-averaged) Runge–Lenz vector, and thus the dynamical algebra of the Kepler problem, is conserved to leading order in the linear friction coefficient.

In elementary terms, although angular momentum $\vec{L}$ and $\vec{S}$ are constants in the Hamiltonian system for $V(r) \sim \frac{1}{r}$ they evolve under linear damping of (3.1), (3.2) as

$$\dot{\vec{L}} = -\beta\vec{L} \qquad \dot{\vec{S}} = -2\beta\vec{L} \times \vec{p}. \tag{3.10}$$

Note that in the weak damping limit, since $\vec{L}$ is conserved to $\mathcal{O}(\beta^0)$, the second equation time averages to $-2\langle\beta\vec{p}\rangle \times \langle\vec{L}\rangle$. Thus, again we learn that if the damping were strictly linear (β constant) then since $\langle\vec{p}\rangle = 0$, the time average of $\dot{\vec{S}}$ is 0, again indicating that the eccentricity vector would be conserved to leading order. Note also that it is straightforward to integrate the $\vec{L}$ equation explicitly, finding $\vec{L} = \vec{L}_0\,\mathrm{e}^{-\beta t}$ the initial condition $\vec{L}_0$ being identified now a conserved quantity of the dissipative system. We use these results in the next section of this paper to amend the 'textbook' orbital secular evolution equations.

4. The damped Kepler problem

The previous section suggests that (linear-) damped bounded Kepler orbits shrink but retain their aspect ratio and do not precess to leading order in the damping. It is well known that superlinear damping does lead to circularization whereas sublinear damping leads to infall orbits in the Kepler case. So far this begs the questions of whether this generalizes to other central field problems, and, if so, then at what order in the linear damping coefficient do orbits undergo shape and precessional change. In this section we address both questions, first describing a problem that arises using a time-honored pertubative method for treating general perturbing forces in the Kepler problem, and second, generalize the result of the preceding section to a broad class of central field potentials. We then establish in precise terms the fate of Kepler orbits under linear damping.

Consider the usual secular orbital evolution method (called 'the variations of constants') most common in literature on celestial mechanics, for example, in [48] (chapter 11 section 5, page 323, though see also the treatments of nonlinear friction in [49–51]). In the 'variations of constants' method, orbital response to an applied force $\vec{F} = R\vec{x} + N\vec{L} + B\vec{L} \times \vec{x}$, in the orbit's tilt Ω, the orbital plane's axis, i, the eccentricity ϵ, the angle of the ascending node ω, the semi-major axis a and the period $T = 2\pi/n$ (in their notation) evolve following [48],

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{nar}{\sqrt{1-\epsilon^2}} N \frac{\sin u}{\sin i} \tag{4.1}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{nar}{\sqrt{1-\epsilon^2}} N \cos u \tag{4.2}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{na^2\sqrt{1-\epsilon^2}}{\epsilon}\left[-R\cos\theta + B\left(1+\frac{r}{P}\right)\sin\theta\right] - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{4.3}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = na^2\sqrt{1-\epsilon^2}[R\sin\theta + B(\cos\theta + \cos E)] \tag{4.4}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2na^2\left[R\frac{a\epsilon}{\sqrt{1-\epsilon^2}}\sin\theta + B\frac{a^2}{r}\sqrt{1-\epsilon^2}\right] \tag{4.5}$$

and where

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{3n}{2a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{4.6}$$

with $u = \theta + \omega$ and for the unperturbed Kepler orbit, $\frac{P}{r} = 1 + \epsilon\cos\theta$, P is the latus rectum, and E is the anomaly, i.e. $r = P(1 - \epsilon\cos E)$. The central angle θ is found via the usual definition of angular momentum. When we specialize these Kepler orbit evolution equations to the case of isotropic and homogeneous friction we learn that (see [48], chapter 11, section 7 but using $\beta(p)p$ for T in that reference)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2pa^2\beta(p)p \tag{4.7}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{2\sin\theta}{\epsilon}\beta(p) \tag{4.8}$$

and

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 2(\cos\theta + \epsilon)\beta(p). \tag{4.9}$$

We can now specialize further to the marginal case, linear friction $\beta(p) = \beta = \text{const}$. To integrate these equations, note $r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = L = L_0\,\mathrm{e}^{-\beta t}$ and, in terms of the force components, $N = 0$, and $R = \beta(p)p\cos\upsilon$ and $B = \beta(p)p\sin\upsilon$ where υ is the angle between the radius vector and the tangent to the orbit. That angle can be written using the parameteric form of r in terms of the constants of the orbit and the angle θ ($\sin\upsilon = L/rp$ and $\cos\upsilon = \epsilon\sin\theta/Lp$) resulting in a self-contained pair of ODE's in ϵ, θ and t,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{e}^{+3\beta t}}{L_0^3}(1 + \epsilon\cos\theta)^2 \tag{4.10}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -2\beta(\cos\theta + \epsilon). \tag{4.11}$$

If we integrate these to leading order in β only (by, for example, using the first equation to eliminate the time derivative to leading order in β) we do indeed find that the eccentricity is an orbit-averaged constant of the motion. But difficulty arises when we try to understand these equations beyond leading order in the damping, as a direct numerical integration of the equation set reveals (figure 1). For a broad set of initial angles and small initial eccentricities, the ϵ passes through zero and goes negative. For comparison, the eccentricity (i.e. the square root of the length of the $\vec{S}$ vector) computed by numerical integration of the original equations of motion for precisely the same mechanical parameters and initial conditions is included in that figure.

Even if one only wanted to assign importance to the asymptotic change in the eccentricity, that asymptotic change from integrating the equation pair (4.10), (4.11) does not scale correctly with the damping coefficient, as may be checked numerically (see [48] for further admonitions against using the 'variations of constants' method over long timescales). Clearly the 'variations of constants' method at higher orders in the evolution leads to unphysical results at short and long timescales. The fault is traceable to the fact that in higher order there are β- (the damping coefficient) dependent terms in the orbit shape whose contributions are ignored substituting for r using the undamped Kepler shape of the ellipse. This substitution is however inextricably part of the 'variation of constants' method. To further clarify this problem with the 'variations of constants' method, it is not due to some ambiguity in the eccentricity of a non-closed orbit, since eccentricity itself, rendered as the length of the $\vec{S}$ vector, has a local definition. Algebraically, with this definition of the eccentricity, note that $\epsilon^2 - 2L^2U = k^2$ in the $1/r$ potential even under arbitrary damping. A more useful algebraically identical form is $\epsilon^2 = 4\mathcal{V}^2r^2 - 2\mathcal{R}^2H$, from which, since H is negative for any damping function on a bounded orbit, we see immediately that ϵ^2 is bounded away from zero.

We now, in two parts, describe an approach emphasizing the secular evolution of the dynamical symmetry, which addresses this mismatch with the usual 'variation of constants'

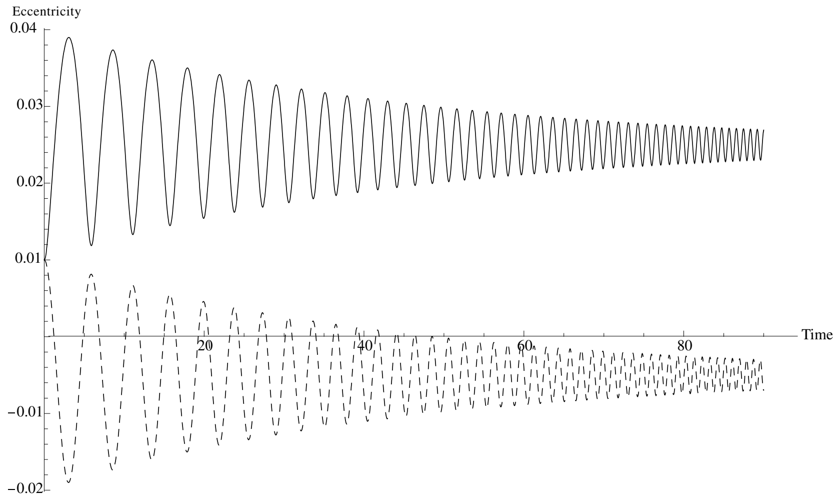

Figure 1. The eccentricity in the actual damped Kepler problem (solid curve) compared with the eccentricity from (4.10) and (4.11) (dashed curve) versus time. Note that for the later the eccentricity can oscillate through zero and can even, as in this case, asymptote to a negative value.

method. For simplicity we focus in the main on potentials with fixed scaling weight α, denied through $V(r) = kr^{\alpha}$. Orbits in any central potential are characterized by a fixed orbital plane and a single dimensionless parameter, the ratio d/c of the perihelion distance d to the aphelion distance c. Let L denote the angular momentum so that $V_{\text{eff}}(r) = \frac{L^2}{2r^2} + V(r)$ is the effective potential. Then from $V_{\text{eff}}(c) = U = V_{\text{eff}}(d)$ where U is the total energy for a $V(r)$ of a fixed α, we have

$$\frac{U}{k} = \frac{c^{\alpha+2} - d^{\alpha+2}}{c^2 - d^2} \qquad \frac{L^2}{2k} = \frac{c^{\alpha} - d^{\alpha}}{c^2 - d^2} c^2 d^2 \tag{4.12}$$

that then can be reduced to a 'dispersion relation' between U and L,

$$\frac{U^{\alpha+2}}{k^2 L^{2\alpha}} = f(d/c) \tag{4.13}$$

where f in this case is a monotonic function on $[0, 1]$. Note also that $f(x) = f(1/x)$. We call d/c the aspect ratio of the orbit (related to the eccentricity in the $\alpha = -1$ case). Thus in leading order (only) in the damping we think of the RHS as a function of the orbital eccentricity only. In applications, the differential form of (4.13) is particularly useful,

$$\left[(\alpha + 2)\frac{\delta U}{\delta L} - 2\alpha \frac{U}{L}\right]\frac{\delta L}{U} = \frac{f'}{f}\delta(d/c). \tag{4.14}$$

Equations of this sort are often written when referring to the secular evolution of orbital system (see, for example, [52] and references therein). As before consider further only damping forces that are isotropic and homogeneous; they can be written in the form from the previous section, $\vec{F}_{\text{drag}} = -\beta(p)\vec{p}$ (to simplify notation we henceforth drop the vector symbol over the p denoting by p both $|p|$ and $\vec{p}$, unambiguous by context). In the limit of weak damping we expect L to be approximately constant so that, time averaging, we arrive at $\langle \delta L \rangle = -\langle \beta(p) \rangle L$ to leading order in $\beta(p)$. Note also that $\delta U = -\beta(p)p^2$ to leading order in β.

Note that the time derivative of $\mathcal{R}$ is the sum of a Poisson bracket with H plus a term proportional to β (see equation (4.16)). This is, $\frac{\text{d}\mathcal{R}}{\text{dt}} = 2\mathcal{V} + \mathcal{O}(\beta)$ which, averaged over bounded orbits, indicates (the virial theorem) that $\langle \mathcal{V} \rangle = \mathcal{O}(\beta)$. Thus for $V(r) = kr^{\alpha}$ this implies that $\langle p^2 \rangle = \alpha \langle V \rangle + \mathcal{O}(\beta)$ so that $\langle U \rangle = \frac{\alpha+2}{2} \langle V \rangle + \mathcal{O}(\beta)$, which to leading order in β in equation (4.14) indicates

$$-\frac{2\alpha}{\langle p^2 \rangle} (\langle \beta(p) p^2 \rangle - \langle \beta(p) \rangle \langle p^2 \rangle) = \frac{f'}{f} \delta(d/c). \tag{4.15}$$

Thus restricted to linear damping (β constant), but for any α, the averages in (4.15) factorize trivially and the aspect ratio is unchanged to leading order under linear damping. Equation (4.15) also indicates that this will, in general, not be the case for a velocity-dependent damping coefficient. Although for potentials with a fixed scaling exponent α there is but one dimensionless parameter (see the LHS of equation (4.13)), the introduction of the damping coefficient β introduces new length and timescales, indicating that the orbital aspect ratio d/c may be a function of β and time. The fact that β is time odd does apparently not preclude its inclusion to linear order in the orbital aspect ratio in general. Thus, we repeat the conclusion that for any monomial potentials linear damping preserves the orbital shape is not a consequence of dimensional analysis and discrete symmetries.

As a final check, note that relation equations (4.15) and (3.9) are both consistent with the attractors of the secular flow in the orbital shape. For circular orbits p^2 is a constant of the motion (again to leading order in β) and thus the LHS of equation (4.15) is zero, as expected by symmetry. Note that in contrast to (4.15) in equation (3.9) the change in the eccentricity is proportional to the eccentricity for any $\beta(p)$, and since the eccentricity vanishes in this limit its orbit-averaged change by (3.9) does as well. Also the strictly radial infall orbit limit is one in which the inner radius, $d \to 0$, and so the LHS of equation (4.15) being non-zero in this limit looks inconclusive. But, by the definition of f via equation (4.13) we see that in this limit $f \to 0$ or $f \to \infty$ depending on the sign of α. Thus, by (4.12), $L = 0$ and remain zero for any $\beta(p)$. In Tarasov's formulation, since (3.9) is fully vector covariant for isotropic and homogeneous (but otherwise arbitrary $\beta(p)$) the change in the $\vec{S}$ must be along the vector itself for the radial infall case. Furthermore, as indicated in the discussion following (3.9), the change $\Delta\vec{S}$ is linear in $\vec{L}$ (for any $\beta(p)$) which vanishes in the radial infall case. In summary, both prescriptions indicate that circular orbits and radial infall must satisfy $\langle \dot{\vec{S}} \rangle = 0$ for any damping function as expected on the grounds by symmetry.

Furthermore, it is straightforward to go further and perturbatively show using (4.15) and the equations of motion that $\beta = \text{const}$ is the *only* shape-preserving damping function for the Kepler potential ($\alpha = -1$). The result is clearly common to all monomial central potentials only, as it is straightforward to demonstrate a counterexample in a more complicated potential. This is due to the fact that there are no additional length scales in the potential and is not the case with other potentials, such as the effective potential in general relativity (where the Schwarzschild radius arises as a second length scale in the potential).

Returning to the rather general statement (3.9), in the Tarasov formulation, the explicit time dependence of a candidate constant of motion $\mathcal{O}$ gives a second term which cancels the last two terms. If the operator has a fixed momentum scaling weight (for example, L is weight 1 and S is essentially weight 2), the last two terms will be of that same scaling weight only for the case of linear friction, $\beta(p) = \text{const}$. Note that this argument does not rule out the existence of additional constants of the motion in the dissipative system that scale to zero as one goes to the Hamiltonian limit. The argument does, however, certify that in the case of linear friction the original Hamiltonian symmetries do survive to leading order in that friction.

Having shown that the linear friction preserves the eccentricity to leading order begs the question of what happens in higher order in the damping. In the spirit of the discussion after (4.13) where the virial played a key role, consider the time evolution of that part of the dynamical algebra

$$\dot{\mathcal{R}} = \{\mathcal{R}, H\} - \beta(p)\mathcal{R} = 2\mathcal{V} - \beta(p)\mathcal{R} \tag{4.16}$$

$$\dot{\mathcal{V}} = \{\mathcal{V}, H\} - 2\beta(p)(2\mathcal{V} - H) = -\frac{1}{r}\left(\partial_r V + \frac{1}{2}\partial_r(r\partial_r V)\right)\mathcal{R} - 2\beta(p)(2\mathcal{V} - H) \tag{4.17}$$

$$\dot{H} = -2\beta(p)(2\mathcal{V} - H), \tag{4.18}$$

and for completeness, we have

$$\{\mathcal{R}, \mathcal{V}\} = H + \mathcal{V} - V + \frac{r}{2}\partial_r V + \frac{r}{2}\partial_r(r\partial_r V). \tag{4.19}$$

To orient the reader to the content of these, first note the Hamiltonian limit (i.e. $\beta(p) \to 0$ limit) for the Kepler case ($\alpha = -1$), both $\langle\mathcal{V}\rangle \to 0$ and $\langle\mathcal{R}/r^3\rangle \to 0$ as expected. We thus expect both of these time averages to be atleast proportional to some positive power of β. Now, in the absence of damping $\mathcal{V}$ is time even and $\mathcal{R}$ is time odd. Formally, taking β to be time odd preserves this discrete symmetry of the above evolution equations. Since we expect the $\langle\mathcal{V}\rangle$ and $\langle\mathcal{R}\rangle$ to be analytic functions of β, it must thus be that $\langle\mathcal{V}\rangle$ vanishes quadratically as $\beta \to 0$.

An elementary argument now certifies that the $\langle\mathcal{V}\rangle$ must be nonpositive in the damped system. Take $\beta(p) = \beta$ a constant. Consider the radial component of the velocity, $\mathcal{R}/r$. It must average to zero in the $\beta \to 0$ limit. Since the damped orbit must shrink, we thus expect $\langle\mathcal{R}/r\rangle \sim \langle -C\beta\rangle$ for some positive quantity C (a function of the other orbital parameters, etc). But now take the evolution equation (4.16), divide by r and time average. Clearly, integrating by parts, $\langle\dot{\mathcal{R}}/r\rangle = -\langle\mathcal{R}/r^2\dot{r}\rangle = -\langle\mathcal{R}^2/r^3\rangle$ implying that the time average $\langle 2\mathcal{V}/r - \beta\mathcal{R}/r\rangle$ must also be strictly negative. But since $\langle\mathcal{R}/r\rangle$ must already be negative, the $\langle\mathcal{V}\rangle$ must also be strictly negative in the damped system.

Specializing to Kepler ($\alpha = -1$), differentiating $\epsilon^2 = |\vec{S}|^2$ in time and applying the equations of motion of the system with friction, we learn that $\frac{\mathrm{d}}{\mathrm{d}t}\epsilon^2 = -8\beta L^2\mathcal{V}$. Using the fact that $\langle\mathcal{V}\rangle$ is negative and order β^2 and integrating both sides, we learn that the asymptotic change in the eccentricity to leading order is positive and also of order β^2 (note the integral itself scales as $1/\beta$). Furthermore, since these are exact evolution equations, we have shown that the integration is well behaved throughout. Thus linear friction causes Kepler orbits to become more eccentric by a fixed amount that scales with the square of the linear damping coefficient.

5. Conclusion

Typically Hamiltonian symmetries lose their relevance to the geometry of the trajectories when damping forces are added to the Hamiltonian system. If the damping is weak, homogeneous and isotropic, then for linear damping in monomial potentials, we have shown that the orbit-averaged shape is stationary. This can be understood most easily through Tarasov's generalization of conserved quantities from the Hamiltonian context to the non-Hamiltonian setting. This approach also quantifies in precise analytic terms the fate and subsequent utility of the dynamical symmetry algebra in the associated non-Hamiltonian system.

There are three main frameworks for understanding orbital motion in a perturbed central field. The first is directly from the equations of motion; this admits straightforward generalization to the non-Hamiltonian case but somewhat obscures the structure and fate of the dynamical symmetry group. The second, namely the KS construction, embeds the Kepler orbit problem in the higher dimensional set of harmonic oscillators with constraints; this illuminates the dynamical symmetry group but does not seem to readily admit a generalization to the non-Hamiltonian system. Lastly, the geometrical approach, namely that which associates the Kepler–Hamilton equations to geodesic flow on manifolds of constant curvature, also illuminates the dynamical symmetry group while making the generalization to the non-Hamiltonian case somewhat unclear.

In light of these difficulties, we used Tarasov's framework (and applied to the damped central field problem here) for extending Poisson symmetries to dissipative systems, emphasizing its utility in making crisp connections between dynamics, algebra and the geometric character of the solutions. Finally, the dynamical algebra remains whole in first order in the linear dissipative system, but flow at higher order is not trivial. The secular perturbative method 'variation of constants' is not adequate to explain this, however an elementary method based on the virial subalgebra explains the change in the shape of kepler orbits in higher order in linear damping.

Acknowledgments

This work was supported in part by the National Science Foundation through a grant to the Institute of Theoretical Atomic and Molecular Physics at Harvard University and the Smithsonian Astrophysical Observatory, where this work was begun and by a fellowship from the Radcliffe Institute for Advanced Studies where this work was concluded. It is a pleasure to acknowledge useful discussions with Harvard–Smithsonian Center for Astrophysics personell Mike Lecar, Hosein Sadgehpour, Thomas Pohl and Matt Holman.

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