ISCO for Binary Schwartzschild Blackhole System

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1. Binary black hole system are believed to form during galaxy mergers which generally consist of supermassive black holes. An example of galaxy with double core nucleous is NGC 6240.[Komossa et al., 2002]
2. Previous works
Measuring coalescing massive binary black holes with gravitational waves: The impact of spin-induced precession
[Lang and Hughes, 2006]

Variables

1. m,M: Masses of the two blackholes
2. X: Spatial separation between the blackholes as viewed from infinity
3. X_m, X_M: Distance of each masses from common centre of momentum.
4. ω_o: the angular frequency of rotation.

The Schwarzschild metric is

$$g_{\mu\nu} = \text{diag}(A, -\frac{1}{A}, -r^2, -r^2 Sin^2\theta)$$

$$A = 1 - \frac{2m}{R}$$

With this metric, consider the action in terms of the spacetime invariant interval,

$$\mathcal{S} = \int \mathcal{L}\ \mathrm{d}s = \int\ \mathrm{d}s \left(A{u^s}^2 - B{u^r}^2 - r^2{u^\theta}^2 - r^2 \sin^2\theta {u^\phi}^2 \right)$$

Equations of Motion

We get the following equations for time and azimutal coordinates

$$Au^s = \text{const}$$

$$r^2 \sin^2\theta u^\theta = \text{const}$$

And for radial coordinates, we get

$$-2\partial_s(Bu^1) - \partial_r A(u^0)^2 + \partial_r B(u^1)^2 + 2r(u^2)^2 + 2r\sin^2\theta(u^3)^2 = 0$$

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Combining radial and angular equations

In combining the equations we get,

$$-\frac{2L_m}{r_m^2}\partial_\phi\left(\frac{B_M L_m}{r_m^2}\dot{r_m}\right)-\frac{1}{A_M^2}\partial_{r_m}A_M+\frac{\partial_{r_m}B_M L_m^2}{r_m^4}\dot{r_m}^2+\frac{2L_m^2}{r_m^3}=0$$

Where r_m and r_M are the radial coordinates of the smaller and bigger mass respectively.
Now treating other terms except the one in second order as gradient of the effective potential and setting the requirement that $\dot{r}=0$ along with the fact that the potential gradient is zero for stationary orbit gives,

$$-\frac{\partial_{r_m}A_m}{A_m^2}+\frac{2L_m^2}{r_m^3}=0$$

Then for the orbits to be stable the second derrivative of the potential should vanish as well. This after some simplification sets the following condition:

$$r_m - \frac{3}{2}\left(r_m + r_M - 2M\right) = 0$$

In order to test its validity define the following papameters,

$$\begin{aligned} r &= r_m + r_M \\ m &= \tau(m + M) \\ M &= (1-\tau)(m+M) \\ x &= \frac{m+M}{r} \end{aligned}$$

The stable orbit conditions in terms of these parameters is,

$$\frac{3}{2} = \frac{(1-\tau)}{(1-\tau) + \tau\left(\frac{1-2(1-\tau)x}{1-2\tau x}\right)^2} + 3(1-\tau)x \qquad (1)$$

Graphing x va τ

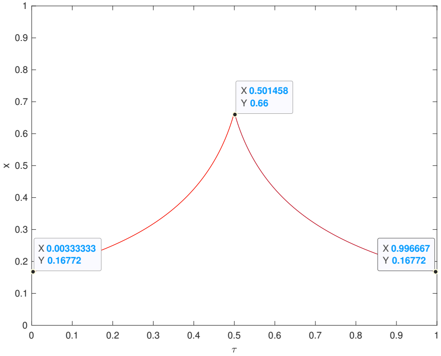

Figure: x vs τ

ISCO for a system with $M >> m$ is at r=6M. Similary, for equal masses the ISCO radius becomes $r = \frac{3}{2}M$

Fomulation

Transformation into cartesian coordinates

The matrix for transforming the space coordinates into cartesian is,

$$\Lambda = \frac{d\{r, \theta, \phi\}}{d\{x, y, z\}}$$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} \\ 0 & \frac{1}{r} & 0 \end{bmatrix}$$

Thus the metric takes following form,

$$g'_{\mu\nu} = g_{\alpha\beta}\Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu} = \text{diag}(A, -\frac{1}{A}, -1, -1)$$

Define

$$u^i = \frac{dx^i}{ds}, \quad i = s, x, y, z$$

In the new coordinates, the action in terms of the space time invariant is,

$$S = \int \mathcal{L} ds = \int (A{u^s}^2 - \frac{1}{A}{u^x}^2 - {u^y}^2 - {u^z}^2) ds$$

$$u^z = 0 \quad [\because \text{there is no motion along z-direction}]$$

Whole Action Approach

Let m and M be the masses of the two blackholes in the system. Then the comnbined action can be written as,

$$\tilde{S} = \int \tilde{\mathcal{L}} dt \qquad = \int (\mathcal{L}_m + \mathcal{L}_M) dt$$

$$= \int \Big[\quad M(A_m {u_M^s}^2 - \frac{1}{A_m} {u_M^x}^2 - X_M^2 {u_M^y}^2) ds_M +$$

$$m(A_M {u_m^s}^2 - \frac{1}{A_M} {u_m^x}^2 - X_m^2 {u_m^y}^2) ds_m \Big]$$

$$= \int \Big[\quad M(A_m {u_M^s}^2 - \frac{1}{A_m} {u_M^x}^2 - X_M^2 {u_M^y}^2) \frac{ds_M}{dt} +$$

$$m(A_M {u_m^s}^2 - \frac{1}{A_M} {u_m^x}^2 - X_m^2 {u_m^y}^2) \frac{ds_m}{dt} \Big] dt$$

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Equations of motion

Imposing Euler Lagrange condition for optimal action gives us equations of motion.

$$i = y : L = (MX_M^2 \frac{dt}{ds_M} + mX_M^2 \frac{dt}{ds_m})\omega_o \qquad (2)$$

$$i = x : \sum_{i=m,M} \left(\frac{\partial}{\partial s_i} \left(\frac{\delta \tilde{\mathcal{L}}}{\delta u_i^x} \right) - \frac{\delta \tilde{\mathcal{L}}}{\delta X_i} \right) \frac{\delta X_i}{\delta X} \frac{ds_i}{dt} \qquad (3)$$

Also the local coordinate time and the assymptotic time are related as,

$$\frac{dt}{ds_m} = \frac{1}{A_m} \text{ and } \frac{dt}{ds_M} = \frac{1}{A_M}$$

This can be obtained using invariant spacetime interval.

Common center of momentum condition

In order for the system to have a common center of momentum, the following should be true,

$$\frac{\delta \tilde{\mathcal{L}}}{\delta \dot{y}} = 0$$

Imposing this condition on the equations of motion gives,

$$\frac{MX_M}{A_m} = \frac{mX_m}{A_M} \qquad (4)$$

However, imposing the condition for stable stationary orbits and combining (2), (3) and (4) gives a complicated implicit relation between the mass and ISCO radius which increases the computational difficulty of the problem.

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Conclusion

For future works,

1. we can look for ways to solve the given equations to find confirm that this approach indeed leas to the expected ISCO condition.
2. we can go further to study the situation of inspiraling, emission of gravitational waves and the rate of energy dissipated in relation to the mass of the black holes.

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Discovery of a binary active galactic nucleus in the ultraluminous infrared galaxy NGC 6240 using ichandra/i.
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