Symmetry Groups in Physics

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Abstract

The study of continuous groups, Lie algebras and their physical significance is of particular interest in theoretical physics, where their existence and structure can be used to explain in a concise and illuminating way the properties of, and connections between, various physical phenomena. However, at the undergraduate level, these structures and their significance is not commonly examined, though they tie together many ideas that can appear in undergraduate classes.

In this paper, the general mathematical framework of symmetry groups and the properties of Lie algebras will be presented along with some basic applications in Quantum Mechanics as well as Classical Physics.

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1 Mathematical Overview

1.1 Groups

Group theory has been called the language of symmetry, and with good reason. It represents a powerful framework with which to abstractly discuss about the symmetries of a system without necessarily referring to the system in question. In the case of groups with a continuum of elements, we can further abstract the workings of the group away by examining elements near the group identity and their relationships. These structures are referred to as Lie Algebras, and are objects of great utility in understanding physics. The reason is simple: by having a language with which to discuss the symmetry properties of a system, one can avoid the overwhelming tangle that solutions of equations can become, or, indeed, avoid the need to solve the equations at all, and still discuss relevant properties of the system.

In the following sections, the defining properties of finite and continuous groups will be presented, not for the sake of a rigorous presentation, (which can and does fill many volumes of textbooks) but rather for the sake of presenting the general principles so that examples can be better understood.

1.1.1 Finite Groups

Though a complete introduction to the theory of finite groups is beyond the scope of this paper, a basic foundation in finite groups is helpful for understanding more complicated structures. Therefore, some fundamental definitions will be presented without discussion.

Definition 1.1.1 *A group is a nonempty set G together with a binary operation $*$ such that*
(1) $a*b \in G$ *for all* $a, b \in G$.
(2) $a*(b*c) = (a*b)*c$ *for all* $a, b, c \in G$.
(3) $\exists\, e \in G$ *such that* $e*a = a*e = a$ *for all* $a \in G$.
(4) $\forall\, a \in G$, $\exists\, a^{-1} \in G$ *such that* $a*a^{-1} = a^{-1}*a = e$.
For $a * b$ *we write* ab *and for* e *we write* 1.

Definition 1.1.2 *The order of a group G, denoted $|G|$, is the number of elements in the set G.*

Definition 1.1.3 *A group is called abelian if* $\forall\, a,b \in G,\ ab = ba$

Definition 1.1.4 *A subgroup of a group G is a nonempty subset H contained in G such that H is a group. In this case we write $H \leq G$.*

Definition 1.1.5 *A function $f : G \to G'$ is called a(n)*
homomorphism *if $\forall a, b \in G,\ f(ab) = [f(a)f(b)]$*
monomorphism *if $f^{\leftarrow}(\{1\}) = \{1\}$*
epimorphism *if $f^{\rightarrow}(G) = G'$*
isomorphism *if it is both a monomorphism and an epimorphism*

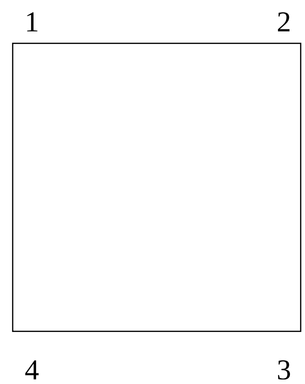
Examples of finite groups include $\mathbb{Z}_m$, the integers from 0 to $(m-1)$ under addition modulo m. Groups of this form are abelian (from the commutivity of addition) and have the property that any proper subgroup (i.e., a subgroup that is not the trivial subgroup $\{1\}$ or the entire group) is generated by an element q which has a non-trivial greatest common factor with m. The subgroup is the collection $\{1, q^1, q^2, q^3, \ldots, q^{r-1}\}$ (where r is the smallest number such that $q^r = 1$ and is called the *order of q* All permutations of the numbers 1 through n form a non-abelian group. Members of this class of group are denoted S_n, and all permutations which involve even numbers of primitive permutations (a primitive permutation is the switching of exactly two elements with each other) form the subgroup A_n.

Let us see an example of how a group can be linked with the symmetries of an object. Consider a square with corners labeled as indicated in Figure 1(a)

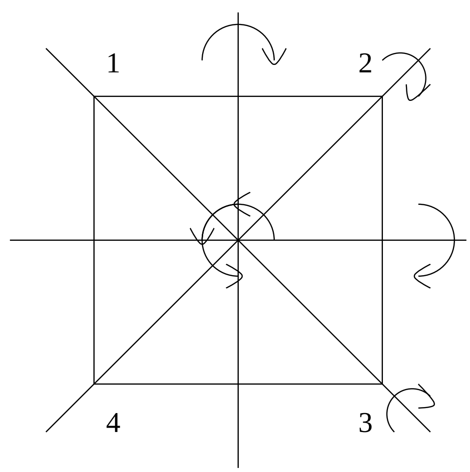
Now, let us consider what geometric transformations can be performed on the square that leaves it unchanged, except for rearranging the labels. It is clear that we can rotate by 90°, 180° or 270°, reflect about the diagonals, or reflect around a line joining the bisectors of opposing sides (Figure 1(b). These operations form a group, since any combination of them still preserves the square, and the operations are associative. This is called the Dihedral group of order 8. Note that this group is not abelian. It can be shown in general that the symmetry group of any regular polygon with n sides is of order $2n$. [1]

1.1.2 Continuous (Lie) Groups

A natural thing to consider is the extension of finite groups to the case where the order of the group is infinite. These groups are refered to as Lie groups, and often arise in discussing the continuous symmetries of objects.



(a) Mapping of Square to Permutation of 4 Elements



(b) Elements of Dihedral Group Relative to Square

Figure 1: Symmetries of a Square

For example, $SO(3)$ is the group of symmetries of the sphere, i.e. all possible rotations in three dimensions. The most common method for understanding these groups is by taking the limit of very small group actions (small meaning arbitrarily near to the identity). In this case, we see that there are usually a finite number of generators, or directions which form a vector space. The study of this space near the identity is the study of Lie Algebras.

1.2 Lie Algebras

The work of Sophus Lie was to show that continuous groups could be studied by linearizing them, and studying the elements of the group that were near the identity. This collection of objects, with the Lie bracket as an operation, form what is called a Lie Algebra. The elements of these algebras are said to generate the group, because any group element can be reproduced as an application of an infinite series of these infinitesimal transformations.

Note that in the following, definitions are from [2]

Definition 1.2.1 *A vector space* L *over a field* $\mathbb{F}$*, with an operation* $L \times L \to L$*, denoted* $[xy]$ *and called the* Lie bracket *or* commutator *or* x *and* y*, is called a* **Lie algebra** *over* $\mathbb{F}$ *if the following hold:*
(L1) The commutator is bilinear.
(L2) $[xx] = 0 \quad \forall x \in L$
(L3) $[x[yz]] + [y[zx]] + [z[xy]] = 0 \quad \forall x, y, z \in L$

Note that Lie algebras are defined in the abstract without reference to continuous groups. That connection will be made clearer later in the context of examples from quantum mechanics, but starting with their definition independent of groups is the standard treatment.

The following definitions are similar to counterparts in the study of groups:

Definition 1.2.2 *A Lie algebra is called abelian if all the commutators are trivial, that is,* $\forall x, y \in L, [xy] = 0$

This follows naturally from the idea of abelian groups. L being abelian implies that $\forall x, y \in L, [xy] = [yx]$. But $\forall x, y \in L, [xy] = -[yx]$ by the definition of the Lie bracket. The only way for these statements to be simultaneously satisfied is $\forall x, y \in L, [xy] = 0$.

Definition 1.2.3 *A linear transformation* $f : L \to L'$ *is called a(n)*
homomorphism *if* $\forall x, y \in L,\ f([xy]) = [f(x)f(y)]$
monomorphism *if* $f^{\leftarrow}(\{0\}) = \{0\}$, *i.e. Ker* $f = 0$
epimorphism *if* $f^{\rightarrow}(L) = L'$
isomorphism *if it is both a monomorphism and an epimorphism*

Note that these are essentially identical to the ideas of homomorphism, monomorphism, epimorphism and isomorphism as expressed in group theory.

Definition 1.2.4 *A subspace* K *of* L *is called a* **subalgebra** *if it is closed under the bracket, that is,* $\forall x, y \in K,\ [xy] \in K$

Definition 1.2.5 *A subalgebra* I *of* L *is called an* **ideal** *if*
$\forall x \in I, \forall y \in L,\ [xy] \in I$
If L *has no such subalgebras, except for 0 and* L*, then we call* L **simple**.

The last has many of the same properties of a normal subgroup N of a group G (usually written $N \trianglelefteq G$). Just as we can create the subgroup G/N so can we create an analogous structure L/I. The geometric picture of L/I is the elements of L projected into all of the directions in the vector space perpendicular to I.
Ideals become important because for the most part we will be interested in the properties of *semisimple* Lie algebras, which will be defined shortly.

Definition 1.2.6 *A* **representation** *of* L *is a homomorphism* $\phi : L \to \mathfrak{gl}(V)$, *where* $\mathfrak{gl}(V)$ *is* End V *as a lie algebra.*

An important representation which exists for every group is called the *adjoint* representation. It is defined by:

$$\forall x \in L,\ \mathrm{ad}x(y) = [xy] \forall y \in L$$

In the case that we wish to treat x only as a member of a subalgebra K, we use the notation ad_K to denote the adjoint restricted to the subspace K.
A similar notation is used for the restriction of the Killing form, defined below, to a subspace.

Definition 1.2.7 *Define* $\kappa : L \times L \to \mathbb{R}$ *by* $\kappa(x, y) = Tr(adx, ady)$. *We call* κ *the Killing form.*

Then κ_K has the simple interpretation of a trace over the directions that lie in K.

1.2.1 Types of Lie Algebras

In this section, we will explore the different general types of Lie algebras, and some of their properties. Specifically, we will look at solvable, nilpotent, simple and semisimple Lie algebras. Again, material is covered as it appears in [2] unless otherwise noted.

Next, we start by defining the *derived algebra* of L by

$$[LL] \equiv \{[xy] \mid x, y \in L\}$$

note that $[LL]$ is an ideal of L (since L is closed, any element of $[LL]$ is also in L, and thus multiplying it by an element of L sends it into $[LL]$ again. Let us extend this to the idea of a *derived series*. First set $L^{(0)} = L$, then define $L^{(n)} = [L^{(n-1)}L^{(n-1)}]$.

Definition 1.2.8 *A Lie algebra L is called* **solvable** *if, for some n, $L^{(n)} = 0$.*

An example of such an algebra would be $\mathfrak{t}(n, \mathbb{R})$ of n-dimensional upper triangular matricies over $\mathbb{R}$. This can be seen by the properties of matrix multiplication for multiplying upper triangular matrices. Let A and B be upper triangular, such that the elements of each look like

$$A_{ij} = \begin{cases} 0, & \text{if } i \geq j \\ a_{ij}, & \text{if } i < j \end{cases} \quad B_{ij} = \begin{cases} 0, & \text{if } i \leq j \\ b_{ij}, & \text{if } i > j \end{cases}$$

let us consider the i, jth element of $[AB]$.

$$\begin{aligned} [AB]_{ij} &= A_{ij}B_{ij} - B_{ij}A_{ij} \\ &= \vec{A}_i \cdot \vec{B}_j - \vec{B}_i \cdot \vec{A}_j \end{aligned}$$

note that the first $i+1$ elements of $\vec{A}_i$ are zero, while the first j elements of $\vec{B}_j$ are the only non-zero ones (if we start counting i and j from 0. Therefore

$$[AB]_{ij} = \begin{cases} 0 & \text{if } i+1 \geq j \\ 1 & \text{if } i+1 < j. \end{cases}$$

Therefore as we continue forming these derived series, eventually the whole matrix becomes zero.

Not only is $\mathfrak{t}(n, \mathbb{F})$ an excellent example of a solvable algebra, it turns out to be the only one:

Theorem 1.2.1 *(***Lie's Theorem***) Let L be a solvable subalgebra of* $\mathfrak{gl}(V)$*,* $\dim(V) < \infty$*. Then the matrices relative to some basis of V are upper triangular.*

The proof of this is beyond the scope of this paper, but can be found in [2].

Define next the *descending central series* by $L^0 = L$, $L^n = [LL^{n-1}]$ Then, similarly to the idea of a solvable algebra, we have

Definition 1.2.9 *A Lie algebra L is called* **nilpotent** *if, for some n,* $L^n = 0$*.*

Now, we will define the type of algebra of primary interest to us.

Definition 1.2.10 *A Lie algebra L is called* **semisimple** *if its maximal solvable ideal (denoted (Rad)L) is* 0*.*

Theorem 1.2.2 *Let L be a Lie algebra. Then L is semisimple if and only if its killing form is nondegenerate.*

Again, this will be accepted without proof due to the scope of this paper.

1.2.2 Roots and Root Spaces

Here we will consider a result which is very important in quantum mechanics. In particular it is what will allow us to find the eigenstates of a system. Semisimple Lie algebras are, in general, built up from copies of $\mathfrak{sl}(2)$, and this will allow us to fix a convenient basis (sometimes called the *Cartan basis*).

We start with a semisimple Lie algebra L. Then, since L is semisimple, it contains elements which are semisimple (i.e. diagonal in some basis). Therefore, there exists a subalgebra of semisimple elements. We call such a subalgebra **toral**. We choose the largest such subalgebra (one which is not properly contained in any other), the traditional notation for which is H. Now, it can be shown that any toral subalgebra (and H in particular) is abelian. Thus we can fix a common basis in the adjoint representation in which all of the elements act diagonally. The number of elements in such a subalgebra is called the *rank* of L. [5] [2]

By diagonalizing the adjoint representations of H, the remaining elements satisfy $[h_i x] = \alpha_i x$, and the "α_i"s are non-degenerate, by Cartan's theorem.

We call the "α_i"s the *roots* and the "x"s the *root vectors* [5]. L can then be expressed as a direct sum

$$L = \oplus L_\alpha, \text{ where } L_\alpha = \{x \in L | [h_i x] = \alpha_i x \quad \forall h_i \in H\} \tag{1.1}$$

The root vectors, α, have certain properties of importance, [2]

Theorem 1.2.3 *(R1)* $\alpha \neq 0$ *is a root implies that* dim $L_\alpha = 1$
(R2) If $\alpha \neq 0$ *is a root, then the only multiples of* α*. which are roots are* $\pm\alpha$*.*
(R3) If $\alpha, \beta \neq 0$ *are roots, then* $\beta(h_\alpha) \in \mathbb{Z}$*, and* $\beta - \beta(h_\alpha)$ *is a root.*
(by $\beta(h_\alpha)$ *we mean* $\vec{\beta}^\dagger h_\alpha$*)*
(R4) If $\alpha, \beta \neq 0$ *are roots, and* $\alpha + \beta$ *is a root, then* $[L_\alpha, L_\beta] = L_{\alpha+\beta}$*.*
(R5) Let $\alpha, \beta \neq 0$ *be roots, and* $\beta \neq \pm\alpha$*. Let q, r be the largest integers for which* $\beta - r\alpha$*,* $\beta + q\alpha$ *are roots. Then* $\{\beta + i\alpha | -r \leq i \leq q\}$ *are all roots.*
(R6) L is generated by the root spaces L_α*.*

These facts are of great utility in finding eigenvectors, and an example of the use will be given in section 2.2.3.

2 Quantum Mechanics

Symmetry groups and Lie algebras appear throughout physics, but it is in quantum mechanics that their presence is most obvious. This is because the structure of the theory is such that it can be formulated entirely in a linear algebra context, where we have a vector space V and we are concerned with observables $O \in \text{End}V = \mathfrak{gl}(V)$, which is also the space in which all representations lie. We will begin by reviewing this linear structure of quantum mechanics and then discussing some examples of how Lie algebras arise and are used.

2.1 Linear Algebra Formalism

To understand how Lie groups and algebras arise naturally out of quantum mechanics, we must first understand Dirac's linear algebra formulation of quantum mechanics. We begin by postulating that the state of a system at a given time is described by a complex, square-integrable function $\psi(x)$. Now, a known [3] property of these functions is that they form the vector space L_2. We can therefore refer to them as vectors abstractly by $|\psi\rangle$. On this vector space we have the natural inner-product given by

$$\langle\psi|\psi\rangle = \int_a^b \psi^*(x)\psi(x)dx$$

Incidentally, it can also be shown that this space is complete [3]. It will form the foundation of our Hilbert space $\mathcal{H}$, which needs to be expanded to accommodate other properties of particles.
On this vector space we define the linear operators $\hat{X}, \hat{P}$ (position and momentum, respectively) by their action on their respective eigenvectors

$$\begin{aligned} \hat{X}|x\rangle &= x|x\rangle \\ \hat{P}|p\rangle &= p\,|p\rangle \end{aligned} \tag{2.1}$$

where the eigenvectors are subject to

$$\begin{aligned} \langle x|x'\rangle &= \delta(x-x') \\ \langle p|p'\rangle &= \delta(p-p') \end{aligned} \tag{2.2}$$

and $\delta(x)$ is the Dirac delta function:

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \quad \text{such that} \quad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

We begin to see the algebra structure arise when we impose the canonical commutation relation

$$\hat{X}_i\hat{P}_j - \hat{P}_j\hat{X}_i = i\hbar\delta_{ij} \tag{2.3}$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

From here, we can build the quantum analogue of any classical observable via promotion of canonical coordinates and momenta to operators (with appropriate symmetrization, i.e. $xp \Rightarrow \hat{X}\hat{P} + \hat{P}\hat{X}$), keeping in mind that for this observable to represent a measurable quantity, it must be Hermitian (i.e. self-adjoint: $\hat{O}^\dagger = \hat{O}$)

Using these observables, as well as other linear operators acting on $\mathcal{H}$ we will be able to construct Lie algebras with our Lie bracket operation defined as the *commutator:*

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \tag{2.4}$$

It is immediately apparent that this satisfies conditions (L1), (L2), and (L3), as this is simply the natural commutator in End $\mathcal{H}$.

2.2 Space and Time Symmetries

The most natural place for symmetry groups to appear within the context of quantum mechanics are in the nieve symmetries inherent in certain situations, i.e. translational, rotational or time invariance. However, the algebras which generate space and time translational symmetries are not terribly interesting, but will be included for the sake of completeness.

2.2.1 Time Translation Symmetry

Time translations are governed by the propagator, $\mathcal{U}(t)$, which is known to be expressible as

$$\mathcal{U}(t) = e^{-i\frac{\hat{H}}{\hbar}t}$$

so the group of translations through time is $U(1)$, and the generator of the algebra is simply $\hat{H}$. [6]

2.2.2 Translational Symmetry

Let us denote by $\mathcal{T}_a$ an operator which translates the entire physical system by a. The action is most clearly seen on the eigenstates $|x\rangle$

$$\mathcal{T}_a|x\rangle = |x + a\rangle \tag{2.5}$$

such that under $\mathcal{T}_a$: [6]

$$\begin{aligned} \langle\hat{X}\rangle &\Rightarrow \langle\hat{X}\rangle + a \\ \langle\hat{P}\rangle &\Rightarrow \langle\hat{P}\rangle \end{aligned} \tag{2.6}$$

It is clear that if we take two such translations, $\mathcal{T}_a$ and $\mathcal{T}_{a'}$, then applying them one after another is equivalent to $\mathcal{T}_{a+a'}$. Also, since addition is commutative, the set of all such translations form a continuous abelian group. Since it is a continuous group, we can learn a little more about it by looking at the near-identity elements that generate it.

What are the near-identity generators of this group? To find them, we first expand $\mathcal{T}_a$ in a Maclaurin series for small a.

$$\mathcal{T}_a = \mathbb{I} - \frac{ia}{\hbar}\hat{A} + \dots \tag{2.7}$$

where $\hat{A}$ is the generator we wish to find. To find it, we note that since $\mathcal{T}_{\vec{a}}$ is, by definition, a translation, then the following must hold [6]

$$\begin{aligned}
\psi(x-a) &= \langle x|\mathcal{T}_a|\psi\rangle \\
&= \langle x|\psi\rangle - \frac{ia}{\hbar}\langle x|\hat{A}|\psi\rangle \\
&= \psi(x) - a\frac{\partial}{\partial x}\psi(x) \\
\frac{\partial}{\partial x}\psi(x) &= \frac{i}{\hbar}\langle x|\hat{A}|\psi\rangle \\
\frac{i}{\hbar}\langle x|\hat{P}_x|\psi\rangle &= \frac{i}{\hbar}\langle x|\hat{A}|\psi\rangle \\
\hat{P}_x &= \hat{A}
\end{aligned}$$

so the generators of this group are exactly the momentum operators, $\hat{P}_i$, which satisfy the trivial commutation relations

$$[\hat{P}_i, \hat{P}_j] = 0 \qquad \forall\, i, j$$

2.2.3 Rotational Symmetry

Let us denote by $\mathcal{R}_\theta$ an operator which rotates the entire physical system through an angle θ, that is [6]

$$\mathcal{R}_\theta|x, y\rangle = |x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta\rangle \tag{2.8}$$

and similar on the momentum eigenstates. Let us perform two of these rotations in succession:

$$\begin{aligned}\mathcal{R}_\theta\mathcal{R}_\phi|x,y\rangle =&\mathcal{R}_\theta(\mathcal{R}_\phi|x,y\rangle)\\ =&|(x\cos\phi - y\sin\phi)\cos\theta - (x\sin\phi + y\cos\phi)\sin\theta,\\ &\quad(x\cos\phi - y\sin\phi)\sin\theta + (x\sin\phi + y\cos\phi)\cos\theta\rangle\\ =&|x(\cos\phi\cos\theta - \sin\phi\sin\theta) - y(\sin\phi\cos\theta + \cos\phi\sin\theta),\\ &\quad x(\cos\phi\sin\theta + \sin\phi\cos\theta) - y(\sin\phi\sin\theta + \cos\phi\cos\theta)\rangle\\ =&|x\cos\theta+\phi - y\sin\theta+\phi,\, x\sin\theta+\phi + y\cos\theta+\phi\rangle\\ =&\mathcal{R}_{\theta+\phi}|x,y\rangle\end{aligned}$$

so it is closed, and can be shown to be closed in the three dimensional case (though it is quite tedious). As it is an excellent example of how the properties of Lie algebras are used to understand and solve problems in quantum mechanics, the derivation of the angular momentum eigenstates is given below.

We first take as fundamental the commutation relationships, rather than their differential operator realizations [5],

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k \tag{2.9}$$

and we attempt to find eigenstates and eigenvalues for these operators. Since these matrices do not commute with one another, there exists no basis in which they are simultaneously diagonalized. Therefore, we rewrite the algebra by

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

such that

$$\begin{aligned}[\hat{J}_z, \hat{J}_\pm] &= \hbar\hat{J}_\pm\\ [\hat{J}_+, \hat{J}_-] &= 2\hbar\hat{J}_z\end{aligned} \tag{2.10}$$

Now that $\hat{J}_z$ is singled out, we seek a maximal toral subalgebra that we can diagonalize. Since $SO(3)$ is of rank one, there is only one element in it's maximal toral subalgebra, which we take to be J_z. Note that in physical applications with central potentials we wish to diagonalize $\hat{H}, \hat{\mathbf{J}}^2, \hat{J}_z$, (where $\hat{\mathbf{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$), but in general, we can only guarantee that $\hat{\mathbf{J}}^2, \hat{J}_z$ will commute, because $\{J_z\}$ is toral, and $\hat{\mathbf{J}}^2$ is the quadratic casimir.

So we now seek eigenstates and eigenvalues of the form $|\beta, m\rangle$, where β^2 is an eigenvalue for $\hat{\mathbf{J}}^2$ and m is an eigenvalue for $\hat{J}_z$. From their definition, it is clear that $\beta^2 \geq m^2$ [6]. This requirement means that there exists a highest value of m such that

$$\hat{J}_+|\beta, m\rangle = 0$$

We call this state m_0. Then we have

$$\begin{aligned} \hat{J}_-\hat{J}_+|\beta, m_0\rangle &= 0 \\ (\hat{\mathbf{J}}^2 - \hat{J}_z^2 - \hbar\hat{J}_z)|\beta, m_0\rangle &= 0 \\ (\beta^2 - m_0^2 - \hbar m_0^2)|\beta, m_0\rangle &= 0 \\ m_0(m_0 + \hbar) &= \beta^2 \end{aligned}$$

and similarly for the minimum m. The max and min are separated by $2m_0$, therefore the eigenstates take the form

$$|j, m\rangle \quad \text{with } j \in \frac{\mathbb{Z}}{2}, \text{ and } j \geq m \geq -j \tag{2.11}$$

2.3 Rotation Group Representations and Q.M.

In this section it will be shown how, in a reversal of the usual situation, a quantum mechanical system can help provide a better understanding of various dimensional representations of the $SU(2)$ algebra.

Consider the quantum mechanical isotropic harmonic oscillator, with Hamiltonian (expressed in the position basis)

$$\hat{H} = \frac{-\hbar^2}{2\mu}(\partial_x^2 + \partial_y^2) + \frac{\mu\omega_0^2}{2}(x^2 + y^2)$$

this Hamiltonian splits into an x and y hamiltonian, each of which can be split into raising and lowering operators. We define the following:

$$\begin{aligned} \hat{a} &= \tfrac{1}{\sqrt{2}}\left(\sqrt{\tfrac{\hbar}{2\mu}}\partial_x + \sqrt{\tfrac{\mu\omega_0}{2\hbar}}x\right) \\ \hat{a}^\dagger &= \tfrac{1}{\sqrt{2}}\left(-\sqrt{\tfrac{\hbar}{2\mu}}\partial_x + \sqrt{\tfrac{\mu\omega_0}{2\hbar}}x\right) \\ \hat{b} &= \tfrac{1}{\sqrt{2}}\left(\sqrt{\tfrac{\hbar}{2\mu}}\partial_y + \sqrt{\tfrac{\mu\omega_0}{2\hbar}}y\right) \\ \hat{b}^\dagger &= \tfrac{1}{\sqrt{2}}\left(-\sqrt{\tfrac{\hbar}{2\mu}}\partial_y + \sqrt{\tfrac{\mu\omega_0}{2\hbar}}y\right) \end{aligned} \tag{2.12}$$

such that

$$\hat{H} = \hbar\omega_0(\{\hat{a}^\dagger, \hat{a}\} + \{\hat{b}^\dagger, \hat{b}\})$$

where $\{a, b\} = ab + ba$, the anticommutator. Then we can construct the state space of the 2-d harmonic oscillator (figure 2(a)). Once we have the state space, we can construct operators which realize the same algebra as $\hat{\mathbf{J}}^2, \hat{J}_z, \hat{J}_+, \hat{J}_-$. The operators we will use are

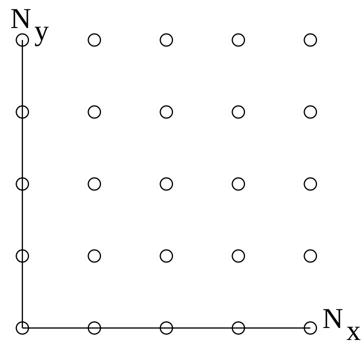
$$\begin{array}{rcl} \hat{J}_z & = & \frac{\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}{2} \\ \hat{J}_+ & = & \hat{a}^\dagger \hat{b} \\ \hat{J}_- & = & \hat{a} \hat{b}^\dagger \\ \hat{\mathbf{J}}^2 & = & \frac{\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}}{2} \left(\frac{\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}}{2} + 1 \right) \end{array} \tag{2.13}$$

where $\frac{\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}}{2}$ gives j, the $\hat{J}_z$ eigenvalue of the highest weight vector (the state on which $\hat{J}_+$ gives 0 identically). The reason for setting up the algebra in this way is because this gives us a simple visual way to understand the n-dimensional representation of the $SU(2)$ spin algebra (see figure 2(b)). Note also that the dimension of the representation is exactly the value $|q - r|$ from theorem 1.2.3. The $J_\pm$ operators move diagonally between eigenstates in the same representation, with the $m = 0$ states lying on the line $N_y = N_x$, while J_z maps each point to itself and gives the weight of the eigenstate relative to the $m = 0$ diagonal.

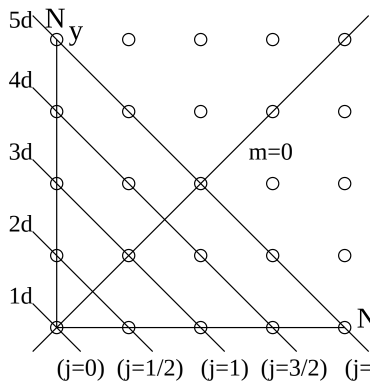
If one has a system of two particles of certain spins, one can find all possible decompositions of the system by starting at the level of the representation of one particle, then add to each lattice point the representation of the other. For example, if adding a spin-1 particle to another, we start at the representation for one and for each lattice point we move two points up, move diagonally up to the same level set, and move two points right. All the points counted once forms one possible result, all points counted twice another, and three times another and so on. In the figure, we add $j = 1$ to $j = \frac{1}{2}$, note the result of $\frac{3}{2} \otimes \frac{1}{2}$. The specific admixture of the possiblities one finds in this matter is given by the Clebsch-Gordon coefficients. Thus we can see that the addition of angular momenta is equivalent to mapping from one N dimensional representation onto another $N + M$ dimensional one, and this mapping may only go onto a subspace of the representation.

3 Classical Mechanics

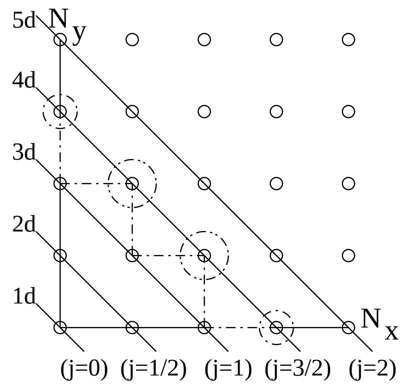
In classical mechanics, the appearance of Lie algebras is not quite as immediately natural as in quantum mechanics, but the structure of Hamiltonian



(a) State space of the 2-d isotropic oscillator



(b) Spin representations on the lattice of the state space



(c) Adding j=1/2 to j=1, note the j=3/2 and j=1/2 subgroups

Figure 2: 2-d Oscillator States and SU(2) Representations

dynamics allows them to appear in a natural fashion nonetheless. In this section, we will review that structure, and again look at examples that demonstrate what the existence of these algebras can tell us about the system in question.

3.1 Hamiltonian Dynamics

Consider a conservative system of particles with N generalized coordinates $\vec{q}$ and their conjugate momenta $\vec{p}$. Then this system is described by some Hamiltonian $\mathscr{H}(q_1..q_N, p_1..p_N)$, and the equations of motion are [4]

$$\begin{aligned} \dot{p}_i &= -\frac{\partial \mathscr{H}}{\partial q_i} \\ \dot{q}_i &= \frac{\partial \mathscr{H}}{\partial p_i} \end{aligned} \tag{3.1}$$

Using these equations of motion, we can find the equations of motion for any given observable $\mathscr{O}(q_1..q_N, p_1..p_N)$:

$$\dot{\mathscr{O}} = \sum_{i=1}^{N} \left(\frac{\partial \mathscr{H}}{\partial q_i} \frac{\partial \mathscr{O}}{\partial p_i} - \frac{\partial \mathscr{O}}{\partial q_i} \frac{\partial \mathscr{H}}{\partial p_i} \right) \tag{3.2}$$

It is easily shown that equation 3.2 gives equations 3.1 in the case that $\mathscr{O}$ is equal to p_i or q_i. That equation 3.2 holds for all observables is a consequence of the chain rule. We now generalize the above operation to the *Poisson bracket*

$$\{A, B\} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right) \tag{3.3}$$

Note that because the derivative is linear, this bracket is bilinear ($L1$), and it is antisymmetric under $A \leftrightarrow B$ ($L2$). It also can be shown that the Jacobi identity holds ($L3$), though demonstrating it is rather involved and lengthy and so will not be done explicitly here. Thus we do indeed have good Lie brackets with which to discuss symmetry algebras in classical mechanics.

3.2 The Appearance of Lie Algebras within Classical Mechanics - Conserved Quantities

Lie algebras come up often in classical mechanics in connection with conserved quantities of the motion (i.e. elements which commute with $\mathscr{H}$.

These tell us about the symmetries of the Hamiltonian, but in fact form only a subalgebra of the full space. For example, in systems in which $\mathscr{H}$ has rotational symmetry, the $SO(3)$ algebra of rotations appears as the vector components of the angular momentum,

$$\begin{aligned} L_x &= yp_z - zp_y \\ L_y &= zp_x - xp_z \\ L_z &= xp_y - yp_x \end{aligned}$$

3.3 An Example (from Work Done at YSU) The SO(3,2) Group of the 2-d Isotropic Harmonic Oscillator

Here we will consider the process of deducing the complete algebra of a physical system (in this case the 2-d isotropic harmonic oscillator) Also, the meaning of the existence of certain subalgebras within the full algebra will be discussed. This work was done in connection with research to be submitted for publication in summer of 2007.

We start out by looking for conserved quantities in the 2-D isotropic harmonic oscillator. We seek quantities quadratic in the cononical coordinates and momenta which commute with the Hamiltonian. This means that we wish to find $\mathscr{O}$ which satisfy (from equation 3.2)

$$\dot{\mathscr{O}} = \{\mathscr{H}, \mathscr{O}\} = 0 \tag{3.4}$$

$$\text{where } \mathscr{H} = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} \tag{3.5}$$

Note that for simplicity we are working in rescaled units to supress constants

$$\begin{aligned} \vec{x} &\rightarrow \sqrt{m}\omega_0 \vec{x} \\ \vec{p} &\rightarrow \tfrac{1}{\sqrt{m}}\vec{p} \end{aligned}$$

Since the potential depends only on $r^2 = x^2 + y^2$, we have immediately that angular momentum $(xp_y - yp_x)$ is conserved. Also, since the hamiltonian factors completely into $\mathscr{H}_x$ and $\mathscr{H}_y$ pieces, the difference $\frac{p_x^2+p_y^2}{2} - \frac{x^2+y^2}{2}$ as well as the sum of these is constant. Now, if we have two constants of the motion A and B

$$\begin{aligned} \{\mathscr{H}, \{A, B\}\} &= -\{A, \{B, \mathscr{H}\}\} - \{B, \{\mathscr{H}, A\}\} \\ &= -\{A, 0\} - \{B, 0\} \\ &= 0 \end{aligned}$$

so, by the Jacobi identity ($L3$), $\{A, B\}$ is also a constant of the motion. Let us take the commutator of the two that we have so far.

$$\left\{xp_y - yp_x, \frac{p_x^2 + p_y^2}{2} - \frac{x^2 + y^2}{2}\right\} = p_x p_y + xy \tag{3.6}$$

We will deonote thse as

$$\begin{aligned} L_z &= xp_y - yp_x \\ L_+ &= p_x p_y + xy \\ L_- &= \frac{p_x^2 + p_y^2}{2} - \frac{x^2 + y^2}{2} \end{aligned}$$

and these satisfy an $SO(3)$ algebra

$$\begin{aligned} \{L_z, L_\pm\} &= \pm 2L_\mp \\ \{L_+, L_-\} &= 2L_z \end{aligned}$$

It turns out that there are no more elements of the algebra which commute with $\mathscr{H}$.

To find the rest of the algebra, we consider quadratic combinations of the coordinates and momenta that have not been considered. One of these combinations is the so called Routhian, which, since one exists for each direction, we take the linear combination $R_\pm = xp_x \pm yp_y$. We note the following, choosing subscripts with an eye towards relationships which will be shown later

$$\begin{aligned} \{\mathscr{H}, R_\pm\} &= -2V_\pm = -2\left(\frac{p_x^2 - x^2}{2} \pm \frac{p_y^2 - y^2}{2}\right) \\ \{\mathscr{H}, V_\pm\} &= 2R_\pm \\ \{V_\pm, R_\pm\} &= -2\mathscr{H} \end{aligned}$$

Where we have identified $\left(\frac{p_x^2 - x^2}{2} \pm \frac{p_y^2 - y^2}{2}\right)$ as the virial. The only other possiblities for elements of the algebra are the cross terms corresponding to these generators, $R_+ = xp_y + yp_x, V_+ = p_x p_y - xy$.

$$\begin{aligned} \{\mathscr{H}, R_+\} &= -2V_+ = -2(p_x p_y - xy) \\ \{\mathscr{H}, V_+\} &= 2R_+ = -2(xp_y + yp_x) \\ \{V_+, R_+\} &= -2\mathscr{H} \end{aligned}$$

This covers all possibilities for generators of the algebra. What can we learn from this structure? First, the fact that the subalgebra commuting with the Hamiltonian is not $SO(2)$ (rotations in two dimensions) as one would expect, but $SO(3)$ (rotations in three dimensions). This is linked to the fact that not only is the potential central, but bound states form closed orbits, and the eccentricity of a given orbit is a conserved quantity.

We can learn even more about the relationship between the generators and the behavior of the system by considering some combinations of the generators:

$$\mathbf{L}^2 = L_z^2 + L_+^2 + L_-^2 = \mathscr{H}^2 \tag{3.7}$$

$$\mathscr{H}^2 - R_z^2 - V_z^2 = L_z^2 \tag{3.8}$$

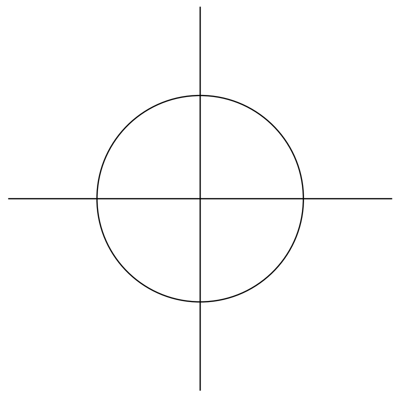
$$\mathscr{H}^2 - R_-^2 - V_-^2 = L_-^2 \tag{3.9}$$

$$\mathscr{H}^2 - R_+^2 - V_+^2 = L_+^2 \tag{3.10}$$

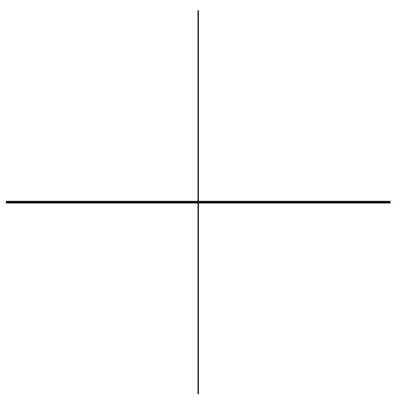
$$2\mathscr{H}^2 - \Sigma R^2 - \Sigma V^2 = 0 \tag{3.11}$$

Where 3.11 is the sum over 3.8, 3.9, and 3.10, simplified using 3.7. We see then from these equations that the action of the $(R, V, \mathscr{H})$ subgroups is that of a cone. The signs change for one commmutator under cyclic permutation of the elements, so this is not $SO(3)$, but $SO(2,1)$. this can also been seen in the quadratic casimir, which is not the sum of the squares, but the sum of R^2 and V^2 minus H^2, again inidicating an $SO(2,1)$ subalgebra. Since $\mathscr{H}$ is the generator of time translations, as time progresses R and V rotate into each other on 2-d slices of these cones. It can be shown that the motion on these cones completely specifies the orbit, and the two free parameters of their phase differences are linked with the two free parameters of the orbit (disregarding arbitrary scalings), that is, the eccentricity and the inclination. In each of the limiting cases of the orbit, one of the cones is at the origin. This is because $\mathscr{H} - L_i^2$ is the radius of of the two dimensional slice of the cone, and in each of the limiting cases, two of the L_is are identically zero for all time, so the square of the remaining one must be equal to the square of the Hamiltonian.

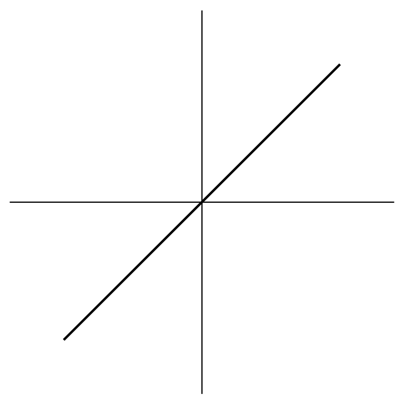
In 3(a), L_+ and L_- are both zero, in 3(b), L_z and L_- are zero, and in 3(c) L_+ and L_z are zero.



(a) $\epsilon = 0$, $\delta = \frac{\pi}{2}$



(b) $\epsilon = 1$, $\delta = 0$



(c) $\epsilon = 1$, $\delta = \frac{pi}{2}$

Figure 3: Limiting Cases of 2-d Oscillator Orbits

4 Gauge Theories

What follows is an outline of how Lie algebras are used to build a physical theory from scratch, rather than arising out of a particular situation in the context of a physical theory. Lie algebras have been used in elementary partical physics to build quantum electrodynamics (a theory built on $U(1)$), quantum chromodynamics (a theory built on $SU(3)$) or the unified electroweak description (a broken $SU(2) \otimes U(1)$. [8] We will build up the theory of a scalar charge particle and an electromagnetism-like field.

We start by noting that quantum mechanically, states only have a phyical meaning up to an overall unitary phase (I.E., under $U(1)$ transformations, the matrix element $\langle\psi|\hat{O}|\rangle$ remains unchanged for all $\hat{O}$). We wish to extend this idea into field theory in a general manner. For the sake of simplicity, we will consider a free scalar field ϕ. To describe the mechanics of this field, we write the Lagrangian

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi \tag{4.1}$$

such that the Euler-Lagrange equations give

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0$$
$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \vec{x}^2} - m^2 \right) \phi = 0$$

Which we recognize as the Klein-Gordon equation, which describes a free relativistic scalar particle.

Since the Lagrangian is quadratic in the field, it is clear that this is invariant under a global rephasing. Let us now require that the Lagrangian remain the same under a spacially dependant rephasing $\phi \to e^{i\alpha(x)}\phi$. Then we have

$$\begin{aligned}
\mathscr{L}' &= \partial_\mu (e^{-i\alpha} \phi^*) \partial^\mu (e^{i\alpha} \phi) + m^2 (e^{-i\alpha} \phi^*)(e^{i\alpha} \phi) \\
&= (-i \partial_\mu \alpha \phi^* + \partial_\mu \phi^*)(i \partial^\mu \alpha \phi + \partial^\mu \phi) + m^2 \phi^* \phi \\
&= \partial_\mu \alpha \partial^\mu \alpha - i(\partial_\mu \alpha) \phi^* \partial^\mu \phi + i(\partial^\mu \alpha) \phi \partial_\mu \phi^* + \mathscr{L}
\end{aligned}$$

So we see that our Lagrangian is not invariant under this transformation. Since breaking of the invariance comes from the Leibnitz property of the derivative, so we introduce an affine term to correct for this [8]

$$D_\mu = \partial_\mu + igA_\mu \tag{4.2}$$

Which transforms in such a way that

$$D_\mu(e^{i\alpha}\phi) = e^{i\alpha} D_\mu \phi \tag{4.3}$$

So we find that

$$\begin{aligned}
e^{i\alpha} D_\mu \phi &= D_\mu(e^{i\alpha}\phi) \\
&= (\partial_\mu + igA_\mu)(e^{i\alpha}\phi) \\
&= (\partial_\mu \alpha)e^{i\alpha}\phi + e^{i\alpha}\partial_\mu \phi + igA_\mu e^{i\alpha}\phi \\
&= e^{i\alpha}\left[\partial_\mu \phi + ig(A_\mu + \frac{1}{g}\partial_\mu \alpha)\phi\right]
\end{aligned}$$

Therefore, Equation 4.2 is satisfied if, under this transformation,

$$A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \alpha \tag{4.4}$$

We identify this field by analogy with classical electromagnetism, where the fields $F_{\mu\nu}$ are not unique to any 4-potential A_μ, but instead are all generated by the family $A_\mu + \partial_\mu \Lambda(x^\mu)$. For this field to be a real degree of freedom, we need to add its derivatives ($F_{\mu\nu} = [D_\mu, D_\nu]$) to the lagrangian

$$\mathscr{L} = D_\mu \phi^* D^\mu \phi + m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{4.5}$$

Note that $F_{\mu\nu}$ as defined above is also invariant under $U(1)$ transformations. These guage fields and their genralizations to non-abelian algebras ($A^a_\mu \tau^a$, where the τ^as are the elements of the algebra) are a very powerful way for understanding the fundamental interactions, and are all built by choosing an affine term to add to the derivative to make it transform nicely. In effect, we choose a symmetry we wish the theory to demonstrate based on observed properties of the interaction field and build up from there.

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