Monographs on Mathematics and Physics

The Mathematics and Physics Behind Electromagnetically Induced Transparency and Power Broadening Using Noise Spectroscopy in Rubidium 87

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Dedication

I dedicate my thesis to my all my colleagues and family for supporting me completely along the way. I would especially like to dedicate my thesis to my father, Bill Snider. I wish he were still here...

Contents

Forward and Preface

This thesis is a requirement of my pursuit of a Bachelor of Science in Mathematics and a Bachelor of Science in Physcis

The author thankfully acknowledges Mary Lynn Savage for mothering me through my undergraduate carreer, Dr. Michael Crescimanno for allowing me the previllage of working on this subjuct with him, Dr. Steven Kent for seeing me through to the end, and both the Mathematics and Physics Departments at YSU for becoming my extended family.

Pure Mathematics is, in its way, the poetry of logical ideas. Albert Einstein

Part I

Introduction

What is mathematics? What can you do with mathematics? These seemingly simple questions do not have simple answers but are very much one in the same question. Mathematics would not have evolved without a need for it. Certain people looked at the world and saw ways to observe quantities and how those quantities change with respect to some other parameter. Simply stated, it was the need to explain our observations of the world that caused mathematics to evolve. In that fashion, it grew into an ever more complex system of tools giving rise to algebra, trigonometry, calculus, and statistics. However, it was all out of the need to understand observations of their universe that lead them to build such powerful tools.

It is through these tools that physics evolved along side mathematics. Mathematics developed the tools that allows physics to explain the world around us. This relationship allowed me the privilege of working on a challenging and complicated project that forced me to implement several mathematical concepts in the process of building an accurate model of Rubidium 87 (^{87}Rb). We use a single continuous wave laser to stimulate and probe the ^{87}Rb cells using the inherent noise in the beam to measure the response at different intensities of the atoms as they excited and de-excite. The resulting linearly polarized light field that was passed through the ^{87}Rb cell then sent through a quarter wave plate to separate the linearly polarized light into a left and right circularly polarized light. The left and right circularly polarized light field was then separated from each other using a beam splitter and sent into separate photodetectors giving data on the response of the intensity of the beam as it encounters the rubidium atoms in the cell. It is this inherently complicated observation that lead us to consider developing of a theoretical model to describe the system. It also helped to coordinate an effort to fully understand the experiment and then possibly implement the concept learned into advanced technologies.

The mathematical concepts I employed in developing this theory with Dr. Crescimanno started with simple rate equations that obeyed population relations that are related to basic properties of matrices. The model continued to develop as we saw the need to find a way to represent fluctuations in the intensity in the equations. As I will demonstrate in the Physical Antecedents section, these additions described a picture of a level structure that we call states. It is these states that represent population energy levels in the system of atoms. Investigating this system, we employed the steady state solution where we set the partial derivative of the populations with respect to time to zero and then solved for the population when there is zero rate of change. In the next section, I will delve into the mathematical concepts involved during the process of developing this Rubidium 87 model.

Part II

Mathematical Preliminaries

Chapter 1

Complex Numbers

A complex number is a number of the form $x + iy$, where x and y are real numbers and i is a symbol with the property that $i^2 = -1$. Let z be a complex number; then $z = x + iy$. The **real part** of z, denoted Re z, is x and the **imaginary part** of z, denoted Im z, is y. Two complex numbers are then only equal if their real and imaginary parts equal each other. Therefore, if $a+ib = c+id$, then $a = c$ and $b = d$ must be true. Since complex numbers have this duality to them you can then represent them as pairs of numbers of the form (x, y). This is a recognizable form from geometry on the real plane thus inferring that there is also a version of the complex plane that is described by a real number and imaginary number that together make up a complex number. Thus the complex plane is created that mirrors the familiar $x - y$ plane but with real numbers falling along the x-axis and imaginary numbers falling on the y-axis.

There are also the typical operations on complex numbers such as addition, subtraction, multiplication, and division. The sum of complex numbers is the same as vector addition. Let $a + ib$ and $c + id$ be two complex numbers. Then their sum is

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

Notice that you just add the real parts and the imaginary parts of both complex numbers. If we now represent those complex numbers as ordered pairs of the form (a, b) and (c, d) and then find their sum, $(a + c, b + d)$, it becomes even clearer that complex addition is just the sum of each part forming a new complex number.

The product of complex numbers depends on the notion that $i^2 = -1$ so that when multiplying the imaginary parts of complex numbers together you get a real number. Let $a+ib$ and $c+id$ be two complex numbers. Then their product is defined as follows

$$\begin{aligned}(a + ib)(c + id) &= a(c + id) + ib(c + id)\\ &= ac + iad + ibc + i^2bd\\ &= ac - bd + iad + ibc\\ &= (ac - bd) + i(ad + bc).\end{aligned}$$

Subtraction of complex numbers is just a combination of the above operations where the idea of multiplying a complex number by a single variable is introduced as well. Let $a+ib$ be a complex number and c be any variable, then $c(a+ib) = ca+icb$. Subtraction uses a special case of this "scalar" multiplication of complex numbers where $c = -1$ so that $c(a + ib) = -1(a + ib) = -a - ib$. Subtraction of complex numbers is then just a combination of this special case and the addition operation.

$$\begin{aligned}(a + ib) - (c + id) &= (a + ib) + (-1)(c + id)\\ &= (a + (-c)) + i(b + (-d))\\ &= (a - c) + i(b - d)\end{aligned}$$

Finding the quotient of complex numbers requires the introduction of the conjugate. Let $z = a + ib$ be a complex number, then the *conjugate* of z is $\overline{z} = a - ib$. From this definition we can deduce some properties for the conjugate of complex numbers letting z and w be complex numbers.

1. $\overline{\overline{z}} = z$
2. $\overline{z + w} = \overline{z} + \overline{w}$
3. $\overline{(zw)} = \overline{z}\,\overline{w}$
4. $\overline{z} = z$ if and only if z is real
5. $\overline{z^n} = \overline{z}^n$ for any integer n
6. $|\overline{z}| = |z|$
7. $|z|^2 = z\overline{z} = \overline{z}z$
8. $|zw| = |z||w|$
9. $|z + w| \leq |z| + |w|$
10. $z^{-1} = \frac{\overline{z}}{z^2}$ if z is non-zero
11. $|\frac{1}{z}| = \frac{1}{|z|}$
12. $e^{\overline{z}} = \overline{e^z}$
13. $\log \overline{z} = \overline{\log z}$

(Brown 2009)

The conjugate can then be used to find the quotient of complex numbers by representing the quotient as a fraction and then multiplying the numerator and denominator by the conjugate of the denominator. This is because $z\overline{z} = a^2 - i^2b^2 = a^2 + b^2$. Thus the quotient of two complex numbers $z = a + ib$ and $w = c + id$ is

$$\begin{aligned}\frac{z}{w} &= \frac{a + ib}{c + id} \\ &= \frac{(a + ib)(c - id)}{(c + id)(c - id)} \\ &= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} \\ &= \frac{z\overline{w}}{|w|^2}.\end{aligned}$$

As noted earlier, a complex number $z = x+iy$ can be represented geometrically by a point (x, y) on the complex plane. However, this can be expressed in polar coordinates, (r, Θ), by letting

$$x = r\cos\theta \text{ and } y = r\sin\theta,$$

so

$$\begin{aligned} z = x + iy &= r\cos\theta + i(r\sin\theta) \\ &= r(\cos\theta + i\sin\theta), \end{aligned}$$

where $r = |z| = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$. The angle θ is called the *argument* of z and is denoted arg z. The arg z is not unique though the *principal argument* of z, denoted Arg z, is since it is defined on an interval from $-\pi < \theta \leq \pi$.

This leads us to a theorem developed by the French mathematician Abraham De Moivre. De Moivre's Theorem states that if $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta),$$

or, in different terms,

$$|z^n| = |z|^n \text{ and } \arg(z^n) = (n)\arg(z).$$

(Brown 2009)

The final and most important aspect of complex numbers is due to Leonhard Euler. It is known from calculus that there are power series expansions for the exponential, cosine, and sine. That is,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Letting x be a real number and $z = ix$ then

$$e^z = e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots.$$

Then using the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on repeating this pattern in a cycle of length 4 we then find that

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots \\ &= (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots) \\ &= \cos x + i\sin x. \end{aligned}$$

Thus Euler's formula simply states that for any real number x,

$$e^{ix} = \cos x + i \sin x.$$

Using Euler's Formula we can write complex numbers in another form

$$z = r(\cos\theta + i \sin\theta) = re^{i\theta}.$$

The conjugate can also be rewritten using the trigonometric identities

$$\cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta,$$

and the definition of the conjugate, $\overline{z} = r(\cos\theta - i\sin\theta)$. Thus, a formula for $\overline{z}$ is derived.

$$\overline{z} = r(\cos(-\theta) + i\sin(-\theta)) = re^{i(-\theta)} = re^{-i\theta}$$

Chapter 2

Linear Algebra

Let us now introduce the concept of linear algebra, which is the study of vector spaces, linear transformations, and matrices. All three of these basic precepts of linear algebra were used to develop the equations that modeled the ^{87}Rb transitions using a noisy continuous wave laser and room temperature. We will begin with a description of a typical elementary algebra problem then move into vector spaces. Continuing from there we will describe linear transformations that lead into matrices.

Understanding any subject should begin with the simplest of concepts and then build upon those ideas. Geometrically we are all familiar with the general concept of a line and its equation form

$$y = mx + b.$$

This is just a linear equation that has been solved for a variable, namely y. Rewriting this equation into another form we get

$$y - mx = b.$$

A system of linear equations is easier to see if the above form is utilized . I will start with an elementary example of a system of two equations with two unknowns.

$$3x - 2y = 1 \tag{1}$$

$$x + y = 2 \tag{2}$$

Since we have a system of two equation with two unknowns we can solve the system explicitly by taking one of the equations and solving for one of the variables in terms of the the other. Namely, we take one equation and put it into the familiar equation of a line that was introduced earlier. We will solve the second equation for x then substitute it into the first equation to get a solution for y.

$$x = 2 - y$$

$$3(2 - y) - 2y = 1$$

$$6 - 1 = 3y + 2y$$

$$5 = 5y$$

$$y = 1$$

Simply substituting this solution into either equation (1) or (2) gives a result for x. We will take the second equation and find a solution for x.

$$x + 1 = 2$$

$$x = 1$$

This results in an explicit solution for the system in terms of x and y. This solution can be written as $(1, 1)$ which is the description of a point in a two dimensional plane. Geometrically we now can see that this system of equations represents two lines that intersect at some point in the plane, namely $(1, 1)$. By solving this system we were able to find that point of intersection of these lines which can only be a single solution since we are dealing with nonparallel straight lines in 2 dimensions.

There are other techniques one could have used to solve the above system of equations. The one that comes to mind and is most common is finding a constant to multiply one equation by so that when you add the equations one variable cancels out leaving you able to find a solution for a single variable. Then you would simply substitute that solution into either of the original equations and then solve for the other variable giving a complete solution to the system of equations.

Now we will complicate things more and move on to an example of where an explicit solution is impossible since we will have two equations and three unknowns. Take the system of the equations

$$x + 3y - z = 0 \tag{3}$$

$$3x + y + z = 1, \tag{4}$$

which are the familiar forms of the equations of planes when thought about geometrically. This system can not be solved explicitly but we can find an explicit solution at a particular value of z, say $z = 0$. Then we would have a simple system like the earlier one for which we can find explicit solutions for x and y. Thus, for $z = 0$ we get the the equations

$$x + 3y = 0$$

$$3x + y = 1$$

with the resulting solutions $x = \frac{3}{8}$ and $y = -\frac{1}{8}$. Now we have a solution that corresponds to a three dimensional point in 3 dimensions, $(\frac{3}{8}, -\frac{1}{8}, 0)$. However, this only gives us the solution for a single value of z, specifically $z = 0$. If we choose another value of z we get another solution different from any of the other values of z that we could choose. For $z = 2$, we obtain the solution $(-\frac{5}{8}, \frac{7}{8}, 2)$. We can understand this change in the solutions for different values of z if we look at the system again geometrically. We originally saw that the equations described planes in a three dimensional space. Thus, the solutions to the system created by equations (3) and (4) would fall along the intersection of the planes. This intersection forms a line of solutions of which there is an infinite number of them.

We had only slightly complicated matters before we ran into the issue of finding solutions based on arbitrary values of one of the variables. Thus, when we continue to complicate matters by introducing another variable, we will call it t for time, to a system of three dimensional equations we get a way of describing how objects in space change with changing time (i.e. a very simple model of the universe). With this idea of changing time we will briefly touch upon a subject that will be discussed in depth in the next section, the derivative. If we are looking for solutions, we will call them four vectors, that satisfy a system of two $xyzt$ equations and we know that $v = (\alpha, \beta, \gamma, \delta)$ and $v' = (\alpha', \beta', \gamma', \delta')$ are solutions, then the difference in the two four vectors is a solution of the system of homogeneous $xyzt$ equations.

$$v - v' = (\alpha - \alpha', \beta - \beta', \gamma - \gamma', \delta - \delta')$$

However, if $v = (\alpha, \beta, \gamma, \delta)$ and $v' = (\alpha', \beta', \gamma', \delta')$ are solutions to homogeneous $xyzt$ equations, then so are the addition of the four vectors as well as any multiple of the them by some arbitrary number λ.

$$v + v' = (\alpha + \alpha', \beta + \beta', \gamma + \gamma', \delta + \delta')$$

$$\lambda v = (\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta)$$

Also, if $(\eta, \xi, \phi, \theta)$ is an arbitrary solution to the homogeneous system of equations and there is a fixed solution to the nonhomogeneous system of equations $(\alpha, \beta, \gamma, \delta)$, then an arbitrary solution of the nonhomogeneous system is

$$v = (\alpha + \eta, \beta + \xi, \gamma + \phi, \delta + \theta).$$

It can be shown every vector of the above form is a solution to the nonhomogeneous system of equations. Thus we have shown that solutions of equations can be expressed in terms of certain operations on vectors, specifically vector addition and scalar multiplication.

In later sections the concept of a light field is introduced that is just a dense set of photons. This is a much more loose definition of a field compared to the traditional mathematical definition. Though defining a field mathematically allows us to define other useful concepts like vector fields. Essentially, a *field* is a nonempty set consisting of two operations, addition and multiplication, that obey simple laws and conditions. Let Υ be a field and let $\alpha, \beta, \lambda \in \Upsilon$. Then Υ must obey the following:

1. $\alpha + \beta = \beta + \alpha$ and $\lambda\alpha = \alpha\lambda$; (Commutative Laws)
2. $\alpha + (\beta + \lambda) = (\alpha + \beta) + \lambda$ and $(\lambda\alpha)\beta = \lambda(\alpha\beta)$; (Associative Laws)
3. $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$; (Distributive Law)
4. Υ contains an element 0 such that $\beta + 0 = \beta$ for all $\beta \in \Upsilon$;
5. For each $\lambda \in \Upsilon$ there exists an additive inverse $-\lambda \in \Upsilon$ such that $\lambda + (-\lambda) = 0$;
6. Υ contains the multiplicative identity, 1, such that $\beta \times 1 = \beta$ for all $\beta \in \Upsilon$; and
7. All elements in Υ except for 0 have a multiplicative inverse such that if α is a nonzero element of Υ then there exists $\alpha^{-1} \in \Upsilon$ such that $\alpha\alpha^{-1} = 1$.

(Poole 2006)

Linear transformations can rotate and/or reflect your system of equations into a new basis. Thus, they are a function that relates one system of equations to another making them equivalent to each other. In physics, transformations are made to change perspective as in when changing the frame of reference that one is using to observe a situation(i.e. observing the system of atoms from the outside then transforming the system so that you are now observing from inside

the system of atoms). For our purposes in later sections we will be rotating elements of the system called *states* into another system of states that simplifies the equations and draws out physical processes hidden to the observer when in the original frame of reference.

Now that linear transformations and vector spaces have been defined all that is left is to introduce matrices, their operations, and some important properties of matrices. A matrix is a rectangular array of elements that can be thought of as a group of column or row vectors. A system of linear equations can be represented by matrices through matrix operations. Matrix operations are only marginally different from traditional operations with the difference occurring in the matrix multiplication operation.

Matrix operations do not change the basic concepts of addition and multiplication though operations do follow a different form. Addition of matrices is rather straight forward as long as the matrices are of the same dimension. The reason they need to be of the same dimension is that matrix addition is just the addtion of each corresponding element of the matrices. To put it more clearly, let A and B be an $n \times m$ matrices and let $\rho_{ij} \in A$ and $\lambda_{ij} \in B$ where i represents the row and j represents the column of a matrix. Then

$$A + B = [\rho_{ij} + \lambda_{ij}].$$

If A and B were not of the same dimension, then they would not have the same number of rows and columns thus there would be elements from one matrix that would not have an element from the other matrix to add causing the operation to fail.

Now we have a list of properties for matrix addition and scalar multiplication that are as follows. Let A, B, and C be matrices of the same size and let c and d be scalars. Then,

1. $A + B = B + A$; (Commutativity)
2. $(A + B) + C = A + (B + C)$; (Associativity)
3. $A + O = A$; (Addative Identity)
4. $c(A + B) = cA + cB$; (Distributivity)
5. $(c + d)A = cA + dA$; (Distributivity)
6. $c(dA) = (cd)A$; and
7. $\mathbf{1}A = A$.

(Poole 2006)

Matrix multiplication follows along similar lines where the dimension of the matrices involved have to be of a specific type. The exception is when multiplying a matrix by a scalar. In such a situation the scalar just multiplies each element in the matrix rescaling the matrix. Multiplying two matrices requires the preceding matrix to have the same number of columns as the second matrices rows. The order in which you multiply matrices matters greatly because you might be able to multiply two non square matrices one way but to multiply them together the other way would mean you do not have the proper corresponding columns and rows to perform the operation. This leads us to a formal definition of matrix multiplication. If A is an $m \times n$ and B is an $n \times r$ matrix, then the product $C = AB$ is a $m \times r$ matrix. Each element of C is computed as follows:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Thus, if $m \neq r$ then BA would be impossible to calculate.

Now we can construct a list of properties of matrix multiplication to summarize its implications. Let A, B, and C be matrices whose sizes allow for the indicated operations and let k be a scalar.

1. $A(BC) = (AB)C$ (Associativity)
2. $A(B + C) = AB + AC$ (Left Distributivity)
3. $(A + B)C = AC + BC$ (Right Distributivity)
4. $k(AB) = (kA)B = A(kB)$
5. $I_m A = A = AI_n$ if A is $m \times n$ (Multiplicative Identity)

(Poole 2006)

The next matrix operation is called the *transpose*. The transpose is a simple operation to comprehend that greatly complicates and structures systems of linear equations when imposed upon them. The tranpose is simply switching the rows for the columns and the columns for the rows of a matrix. Let A be an $m \times n$ matrix. Then interchanging the rows and columns gives the matrix A^T which is an $n \times m$ matrix. Simply stated using componentwise arguments

$$(A^T)_{ij} = A_{ji} \quad \text{forall } i, j$$

Now we can form properties of the matrix transpose. Let A and B be matrices whose sizes allow for the indicated operations to be performed and let k be a scalar.

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = k(A^T)$
4. $(AB)^T = B^T A^T$
5. $(A^r)^T = (A^T)^r$ for all positive integers r

(Poole 2006)

The idea of symmetry is important in physics and mathematics because it allows for ways to understand more complicated systems. Since we have been building these mathematical concepts to apply to physical behaviors then it is no surprise that matrices can be symmetric. This concept of a symmetric matrix is directly related to the transpose. A square matrix A is *symmetric* if $A^T = A$. Thus, symmetry requires that a matrix have the same elements present in the same locations before and after its transpose.

This leads to two interesting properties of symmetric matrices.

1. If A is a square matrix, then $A + A^T$ is a symmetric matrix.
2. For any matrix A, AA^T and A^TA are symmetric matrices.

Another notion in linear algebra deals with the entries of a matrix that lie along the diagonal. This concept is called the *trace* and it are simply just the sum of the entries along the diagonal of a $n \times n$ matrix. Formally, let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the *trace* of A, denoted $\text{tr}(A)$, is

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

The *inverse* of a matrix is related to I, the identity matrix.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The identity matrix has 0's in every element except along the diagonal of the matrix, which has 1's in those respective positions in the matrix. The *inverse* is related to the identity through the following definition. Let A be an $n \times n$ matrix, then the inverse of A is an $n \times n$ matrix A' that obeys the property

$$AA' = I \text{ and } A'A = I$$

If A' exists, then A is called *invertible*. Interestingly, if A is an invertible matrix, then A' is unique.

Consequently, there is an important implication if A is an invertible $n \times n$ matrix. Let a system of linear equations be given by $A\mathbf{x} = \mathbf{b}$; then it has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$ in $\Re^n$.

This leads to actually computing the inverse of a matrix. To achieve this the concept of a *determinant* is required. First, the determinant and inverse of a 2×2 matrix will be presented with an extension into finding the determinant of matrices larger than 2×2. Let A be a square matrix such that

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix};$$

then A is invertible if the $detA = ad - bc \neq 0$, where $detA$ means the determinant of A. Thus the inverse of A is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If the determinant of A is zero, then A^{-1} does not exist. (Poole 2006)

Finding the determinant of a larger matrix is more involved but follows a pattern making it easy to break into parts. Let A be an $n \times n$ matrix with $n \geq 2$. Also, let a_{ij} represent the individual elements of A. Then the *determinant* of A is the scalar

$$\begin{aligned} \det\ A &= |A| \\ &= a_{11}\det\ A_{11} - a_{12}\det\ A_{12} + \cdots + (-1)^{1+n} a_{1n}\det\ A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det\ A_{1j}. \end{aligned}$$

The det A_{1j} is called a minor and it is typical to combine the minor with its plus or minus sign and thus define the $(i, j) -$ cofactor of A to be

$$C_{ij} = (-1)^{i+j} \det\ A_{ij}.$$

Using this idea of a cofactor the definition of a determinant becomes

$$\det\ A = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

This is referred to as the cofactor expansion along the first row, however, the expansion can be performed on any row or column and the same exact scalar would be obtained. This is the basis of the Laplace Expansion Theorem.

Now we will state the **Fundamental Theorem of Invertible Matrices** with parts that have not been talked about, for completeness sake. Let A be an $n \times n$ matrix. Then the following statements are equivalent.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\Re^n$.

(c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(d) The reduced row echelon form of A is I_n.

(e) A is a product of elementary matrices.

(f) $\text{rank}(A) = n$

(g) $\text{nullity}(A) = 0$

(h) The column vectors of A are linearly independent.

(i) The column vecotrs of A span $\Re^n$.

(j) The column vectors of A form a basis for $\Re^n$.

(k) The row vectors of A are linearly independent.

(l) The row vectors of A span $\Re^n$.

(m) The row vectors of A form a basis for $\Re^n$.

(n) $\det A \neq 0$

(o) 0 is not an eigenvalue of A.

(Poole 2006)

Now that some general properties from this section have been established, final concepts from linear algebra can be understood that are used directly in following sections when modeling the behavior or rubidium atoms. In the previous section the concept of a conjugate in complex numbers was introduced. Earlier in this section the idea of a transpose was explored. Now the notion of the complex transpose can be clearly understood. Let A be a complex matrix, then the *complex transpose* of A is the matrix A^* defined by

$$A^* = \overline{A}^T,$$

where $\overline{A}$ is simply the complex conjugate of A.

There are some properties of complex matrices that allow the construction of properties of the complex transpose of matrices. This is analogous to the properties of the transpose for real matrices. The properties of complex matrices are as follows letting A and B be complex matrices and c be a complex scalar.

(a) $\overline{\overline{A}} = A$

(b) $\overline{A+B} = \overline{A} + \overline{B}$

(c) $\overline{cA} = \overline{c}\overline{A}$

(d) $\overline{AB} = \overline{A}\,\overline{B}$

(e) $(\overline{A})^T = (\overline{A}^T)$

(Poole 2006) Using these properties we can now establish the following properties of complex tranpose. Let A and B be complex matices, and let c be a complex scalar.

(a) $(A^*)^* = A$

(b) $(A+B)^* = A^* + B^*$

(c) $(cA)^* = \overline{c}A^*$

(d) $(AB)^* = B^*A^*$

(Poole 2006)

Complex matrices have an analogous concept to symmetry for real matrices called *Hermitian*. Recall that a real matrix A is symmetric if $A^T = A$. Thus, for a square complex matrix, A, to be *Hermitian* then it must equal its own conjugate transpose.

$$A^* = A$$

Consequently, Hermitian matrices have some important properties. The most telling property at initial glance is that the diagonal elements of a Hermitian matrix are all real entries. The only complex elements in a Hermitian matrix are off diagonal as long as they have a nonzero imaginary part. With that knowledge it can be seen that the eigenvalues of Hermitian matrices must be real and the eigenvectors are orthogonal as well. It can then be shown that if A is a Hermitian matrix then the complex conjugate, A^*, is also a Hermitian matrix. Finally, it has the interesting relation to symmetric matrices that says the set of Hermitian matrices are a vector subspace of the vector space of complex matrices and that the real symmetric matrices are a subspace of the Hermitian matrices.

Hermitian matrices can also be thought of as operators acting on a Hilbert space. In that case they are often called self-adjoint operators on a Hilbert space and are used in quantum mechanics because the eigenvalues of Hermitian matrices are observable, or measurable. This measurement in quantum mechanics leading to distinct eigenvalues is the basis of the idea of states that will introduced fully at the beginning of the Physical Antecedents section.

Chapter 3

Differential Equations

In mathematics and physics, models are developed to understand physical phenomena from free fall of a body to radioactive decay. These models are just equations that often contain derivatives of some unknown function. Such equations are called ***differential equations***. Differential equations can be thought of as the ***rate of change*** of one variable with respect to another variable. Differential equations can be functions of one variable or many variables giving rise to the concept of the ***partial derivative***, ∂, which is the derivative of a function $u(x, y)$ with respect to x holding y fixed then y holding x fixed. The partial derivative allows us to solve for the rates of change of individual variables that comprise a function of multiple variables,

$$\begin{aligned}\frac{\partial u(x,y)}{\partial x} = u_x &= \frac{\partial u}{\partial x} \\ &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} \\ \frac{\partial u(x,y)}{\partial y} = u_y &= \frac{\partial u}{\partial y} \\ &= \lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{h}.\end{aligned}$$

(Nagle 2008)

Some common terminology is required to continue discussing topics in differential equations. If an equation involves the derivative of one variable with respect to another variable, then the former is called the ***dependent variable*** and the latter is the ***independent variable***. ***Coefficients*** are the leading terms multiplying variables in an equation. The ***order*** of a differential equation is the order of the highest-order derivatives present in the equation. A differential equation that has only ordinary derivatives with respect to a single independent variable is called an ***ordinary differential equation*** or ODE. A differential equation involving partial derivatives with respect to more than one independent variable is a ***partial differential equation***. For example,

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0,$$

is a second-order differential equation where t is the independent variable and x is the dependent variable and a and k are coefficients. Also,

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y$$

is a partial differential equation where x and y are independent variables and u is the dependent variable.

There are linear and non-linear differential equations with the linear version being the simplest form since it has the advantage of being more amenable to solution that non-linear ones. A linear differential equation is one in which the

dependent variable and its derivatives appear in additive combinations of their first powers. Or more precisely,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x).$$

(Nagle 2008)

Linear second-order differential equations are going to be important in later sections because they are famously used to describe oscillatory motion of a mass attached to a spring. This becomes relevant later when talking about laser beam characteristics. The best known linear second-order differential equation arises from the Simple Harmonic Oscillator (SHO). I will explore the SHO leading into the Damped Harmonic Oscillator (DHO) and ending with the Forced DHO.

The SHO leads to a linear second-order differential equation that describes the motion of a mass attached to one end of an elastic material of negligible weight (i.e. a spring) with the other end fixed so that motion is constrained to one dimension. Newton's second law ($F = ma$) is a second-order differential equation that describes the oscillatory motion of the mass-spring system neglecting all other contributing quantities. This is possible because acceleration (a) is just a change in velocity (v) which in turn is just a change in position (x). Thus velocity is $v = \frac{dx}{dt}$ which implies that acceleration is $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. This says that acceleration is a second-order derivative with the position x being the dependent variable and time t being the independent variable. Filling out the rest of the SHO requires another equation that describes what force the spring exerts on the mass as it is stretched and compressed. For most springs this force is directly proportional to the displacement, or the position x of the mass. This equation is refered to as Hooke's Law and is written

$$F_{spring} = -kx,$$

where k is a positive constant intrinsic to the spring called the spring constant and the negative sign is present to note the opposign nature of the force. Now the components of the SHO can be brought together giving rise to a complete equation for the SHO that has the following form.

$$m\ddot{x} + kx = 0 \tag{5}$$

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x \tag{6}$$

This implies that the net force (F_{net}) is equal to the restorative force from Hooke's law. A general solution can be found for the SHO using the method of inspired guessing. This method takes into account that at time $t = 0$ we get intial values for position, x_0, and velocity, v_0, that uniquely specify the problem.

$$x(t = 0) \equiv x_0$$

$$\frac{dx}{dt}(t=0) = v_0$$

Using these initial conditions we now notice that oscillatory motion is periodic teaching us that periodic functions would be best to describe oscillatory motion. Cosine and sine are both periodic functions leading to the inspired guess for the solution for the displacement x as a function of time t

$$x(t) = x_0 \cos \omega t + b \sin \omega t.$$

This satisfies the initial condition that $x(t=0) = x_0$ but leaves us to find what ω and b are as well as checking that at $t = 0$ we have $\dot{x} = v_0$. Thus, to find the velocity as a function of time requires taking the derivative of $x(t)$

$$v(t) \equiv \frac{dx}{dt} = -\omega x \sin \omega t + \omega b \cos \omega t.$$

Setting $t = 0$ we find that $v_0 = \omega b$ or

$$b = \frac{v_0}{\omega}.$$

Now that we have a solution for b we can now write the proposed solution as

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t. \tag{7}$$

Now all that is left is taking the second derivative of our proposed solution then comparing it with our original differential equation to verify our guess is correct and find a solution for ω.

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\omega^2 x_0 \cos \omega t - \omega v_0 \sin \omega t \\ &= -\omega^2 (x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t) \\ &= -\omega^2 x \end{aligned}$$

Comparing this to equation (6),

$$\frac{d^2x}{dt^2} = -\omega^2 x = \frac{k}{m} x,$$

it is clear that $-\omega^2 = \frac{k}{m}$, or

$$\omega = \sqrt{\frac{k}{m}}.$$

Though in physics we usually see this solution (eq. (7)) in another form:

$$x(t) = A \cos(\omega t + \phi), \tag{8}$$

where A is a positive constant and ϕ is the *phase* of oscillation. It is well known that $|\cos \theta| \leq 1$, thus leading to the result $|x(t)| \leq A$. Since the magnitude of

$x(t)$ is less than or equal to A, it is easy to conclude that A is the ***amplitude*** of oscillation. Now to show that equations (3) and (4) are equivalent, we use the trigonometric identity:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Setting $\alpha = \omega t$ and $\beta = \phi$, equation (8) can be rewritten as

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi.$$

Thus we can conclude they are equivalent if we identify that

$$x_0 = A\cos\phi$$

and

$$\frac{v_0}{\omega} = -A\sin\phi.$$

The amplitude A can be explored in more depth if we rewrite the above equations solving for $\cos\phi$ and $\sin\phi$:

$$\cos\phi = \frac{x_0}{A} \tag{9}$$

and

$$\sin\phi = -\frac{v_0}{\omega A}. \tag{10}$$

Using the trignometric identity $\cos^2\theta + \sin^2\theta = 1$ it is easy to find

$$\frac{x_0^2}{A^2} + \frac{v_0^2}{\omega^2 A^2} = 1$$

and thus by multiplying both sides of the equation by A^2 then taking the square root of both sides a solution for the amplitude A is found.

$$A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}$$

The phase of oscillation, ϕ, can be found by using eqs. (9) and (10).

$$\tan\phi \equiv \frac{\sin\phi}{\cos\phi} = -\frac{v_0}{\omega\ x_0} \tag{11}$$

$$\phi = -\tan^{-1}\left(\frac{v_0}{\omega\ x}\right)$$

However, since we have the complication that ϕ is not uniquely defined there must be an interval of angles that ϕ can range over that will allow us to find a unique phase angle. This interval ranges from

$$-\pi < \phi \leq \pi,$$

noting that only one end point is included because $-\pi + 2\pi = \pi$.

In the process of finding equation (11) there was some information lost, since changing the sign of both x_0 and v_0 changes the phase angle by π but does not change the value of $\tan\phi$ since $\tan(\phi+\pi) = \tan\phi$. This allows us to use equations (5) and (6) to find conditions on v_0 and x_0 that tells where ϕ falls in the interval $-\pi < \phi \leq \pi$.

$$\begin{aligned} v_0 < 0 &\implies 0 < \phi < \pi \\ v_0 > 0 &\implies -\pi < \phi < 0 \\ v_0 = 0 \text{ and } x_0 > 0 &\implies \phi = 0 \\ v_0 = 0 \text{ and } x_0 < 0 &\implies \phi = \pi \end{aligned}$$

The significance of the phase angle is that it determines how much the displacement, $x(t) = A\cos(\omega t + \phi)$, leads or lags behind as compared to the pure displacement, $x(t) = A\cos\omega t$, when $\phi = 0$. This pure displacement has a maximum at $t = 0$, however, $x(t) = A\cos(\omega t + \phi)$ has a maximum when $\omega t + \phi = 0$ or $t = -\frac{\phi}{\omega}$. If $\phi > 0$, then this maximum occurs at $-t$ before the maximum of the pure displacement and thus the phase of $x(t)$ *leads* to that of the pure displacement. If $\phi < 0$, then the maximum occurs at t after the maximum of the pure displacement and thus the phase of $x(t)$ *lags* that of the pure displacement.

The Damped Harmonic Oscillator (DHO) is much like the SHO with the addition of a dissipative force that is directly proportional to the velocity of the mass in the opposite direction. This dissipative force is a good approximation of the behavior of air resistance and produces another differential equation with an exact solution. In actuality, this is the only type of dissipative force for which the differential equation of motion has an exact solution. Thus the differential equation takes the form:

$$m\ddot{x} + b\dot{x} + kx = 0, \tag{12}$$

where b is the drag coefficent. Again we use the inspired guess method and use $x(t) = e^{\alpha t}$ because this function reproduces itself when differentiated. Taking the first and second derivatives of $x(t)$ and then substituting them back into eq. (12) we will be able to find a value for the constant α:

$$(\alpha^2 + \frac{b}{m}\alpha + \omega_0^2)e^{\alpha t} = 0, \tag{13}$$

where ω_0 is the natural frequency. Since $e^{\alpha}t \neq 0$, then it is clear that

$$\alpha = -\gamma \pm \left(\gamma^2 - \omega_0^2\right)^{\frac{1}{2}}, \tag{14}$$

thus leading us to three cases based on the amount of damping.

The first case is called the **underdamped** case and occurs when $\gamma < \omega_0$. This forces solutions for α to be complex

$$\alpha = -\gamma \pm i(\omega_0^2 - \gamma^2)^{\frac{1}{2}} = -\gamma \pm i\beta,$$

where $\beta_1 = \sqrt{\omega_0^2 - \gamma^2}$. Thus, using Euler's formula we have a solution for $x(t)$:

$$\begin{aligned} x(t) &= e^{(-\gamma \pm i\beta_1)t} \\ &= e^{-\gamma t} e^{\pm \beta_1 t} \\ &= e^{-\gamma t}(\cos \beta_1 t \pm i \sin \beta_1 t). \end{aligned} \tag{15}$$

We then notice that the imaginary and real parts of equation (15) are seperately solutions that satisfy the ordinary differential equation. Thus,

$$x_1(t) = e^{-\gamma t} A \cos \beta_1 t \tag{16}$$

$$x_2(t) = e^{-\gamma t} B \sin \beta_1 t \tag{17}$$

where A and B are arbitrary constants. Constructing a general solution, we see that

$$x(t) = e^{-\gamma t}(A \cos \beta_1 t + B \sin \beta_1 t).$$

Using initial conditions at $t = 0$ for x the value for A can be determined by

$$x(0) = x_0 = A$$

and taking the derivative of $x(t)$ then finding the initial conditions when $t = 0$ the value of B can be determined as follows.

$$\begin{aligned} \dot{x} &= -\gamma A e^{-\gamma t} \cos \beta_1 t - A\beta_1 e^{-\gamma t} \sin \beta_1 t - \gamma B e^{-\gamma t} \sin \beta_1 t + \beta_1 B e^{-\gamma t} \cos \beta_1 t \\ &= -\gamma x_0 e^{-\gamma t} \cos \beta_1 t - x_0 \beta_1 e^{-\gamma t} \sin \beta_1 t - \gamma B e^{-\gamma t} \sin \beta_1 t + \beta_1 B e^{-\gamma t} \cos \beta_1 t \end{aligned}$$

$$\begin{aligned} \dot{x}(0) = v_0 &= -\gamma x_0 + \beta_1 B \\ v_0 + \gamma x_0 &= \beta_1 B \\ B &= \frac{v_0}{\beta_1} + \frac{\gamma x_0}{\beta_1} \end{aligned}$$

This leads to the final form of the solution to the underdamped case

$$x(t) = e^{-\gamma t}\left(x_0 \cos \beta_1 t + \frac{\gamma x)0 + v_0}{\beta_1} \sin \beta_1 t\right)$$

where $\beta_1 = \sqrt{\omega_0^2 - \gamma^2}$ and $\gamma = \dfrac{b}{2m}$.

The second case is the **critically damped** case and occurs when $\gamma = \omega_0$. This is the smoothest and fastest way to get to equilibrium when the amplitude $A = 0$. In this case there is only one real solution for α, namely

$$\alpha = -\omega_0.$$

Thus the solution is just the limit of the underdamped case as β goes to zero. That is,

$$x(t) = e^{-\gamma t}\left[x_0 + (\gamma x_0 + v_0)t\right],$$

with the $(\gamma x_0 + v_0)t$ part being a characteristic of an ordinary differential equation that has equal eigenvalues ($\lambda_1 = \lambda_2$).

The third case is the **overdamped** case and occurs when $\gamma > \omega_0$. This time there are two real solutions to $\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$ that lead to a solution for $x(t)$ that is the sum of two exponentials, where

$$\begin{aligned} x(t) &= e^{-\gamma t}(Ae^{\beta_2 t} + Be^{-\beta_2 t}) \\ &= Ae^{\alpha_+} + Be^{\alpha_-}, \end{aligned}$$

such that $\beta_2 = \sqrt{\gamma^2 - \omega^2}$. The coefficents A and B are determined by the initial conditions when $t = 0$. Thus

$$x(0) = A + B$$

and

$$\dot{x}(0) = A\alpha_+ + B\alpha_-,$$

so that

$$A = x_0 - \frac{\alpha_+ x_0 - v_0}{2\beta_2},$$

$$B = \frac{\alpha_+ x_0 - v_0}{2\beta_2},$$

and

$$A = x_0 - B.$$

Using this information we can show that

$$\begin{aligned} x(t) &= e^{-\gamma t}\left[(x_0 - B)e^{\beta_2 t} + Be^{\beta_2 t}\right] \\ &= e^{-\gamma t}\left(x_0 \cosh \beta_2 t + \frac{\gamma x_0 + v_0}{\beta_2} \sinh \beta_2 t\right). \end{aligned}$$

The Forced DHO has the same differential form except instead of being equal to 0 it is equal to some driving force $F_0 \cos \omega t$. Thus Newton's second law now reads

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos(\omega t + \phi). \tag{18}$$

The solution to this differential equation comes in two parts, a homogeneous solution and a particular solution. The homogeneous solution is just the solution to the DHO

$$x_H(t) = e^{-\gamma t}\left[(x_0 - B)e^{\beta_2 t} + Be^{\beta_2 t}\right].$$

The particular solution is found using the inspired guess method and choosing $x(t) = C\cos(\omega t + \phi_0)$. Thus, taking the appropriate derivatives and substituting into equation (18) we find

$$\begin{aligned} &\Big[D - C[(\omega_0^2 - \omega^2)\cos\phi_0 + 2\gamma\omega\sin\phi_0]\Big]\cos\omega t \\ &-\Big[C[(\omega_0^2 - \omega^2)\sin\phi_0 - 2\gamma\omega\cos\phi_0]\Big]\sin\omega t = 0, \end{aligned} \tag{19}$$

where $D = \frac{F_0}{m}$, ω is the driving frequency, and ϕ_0 is the driving phase. Since $\cos\omega t$ and $\sin\omega t$ are linearly independent functions then they both identically vanish to zero. Hence, solving for ϕ_0 in the part of equation (19) that is multiplied by $\sin\omega t$ we obtain

$$\tan\phi_0 = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

We can then conclude

$$\sin\phi_0 = \frac{2\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2) + 4\gamma^2\omega^2}}$$

$$\cos\phi_0 = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2 - \omega^2) + 4\gamma^2\omega^2}}.$$

Now we just need to find the coefficient C to our solution $x_p(t)$ which we can do by looking at the part of equation (19) that is multiplied by $\cos\omega t$:

$$D - C[(\omega_0^2 - \omega^2)\cos\phi_0 + 2\gamma\omega\sin\phi_0] = 0$$

$$\begin{aligned} C &= \frac{D}{(\omega_0^2 - \omega^2)\cos\phi_0 + 2\gamma\omega\sin\phi_0} \\ &= \frac{D}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}. \end{aligned}$$

Thus our particular solution is

$$x_p(t) = \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\cos(\omega t + \phi_0), \tag{20}$$

with driving phase

$$\phi_0 = \tan^{-1}\frac{2\gamma\omega}{\omega_0^2 - \omega^2}.$$

The general solution is now apparent:

$$\begin{aligned} x(t) &= x_H(t) + x_p(t) \\ &= e^{-\gamma t}\left[(x_0 - B)e^{\beta_2 t} + Be^{\beta_2 t}\right] + \frac{F_0}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}\cos(\omega t + \phi_0). \end{aligned}$$

Part III

Physical Antecedents

Chapter 4

Quantum States

The notion of ***states*** in quantum mechanics grows from the need to describe electron orbits in more detail than classical mechanics can provide. Atoms have different states that correspond to the different energy levels which in turn depend on the orientation of the electron as it orbits the nucleus. These different electron orbits also give a shape to atoms that at times can be extremely simple as in a sphere or much more complicated when dipole lobes start to appear. For the purposes of our research, states are generalized and can be thought of as merely energy levels into which atoms are excited or decay. This generalization from electron excitation to atomic excitation is easy to see. Once an external field is applied to the system of atoms we can then generalize the excitation of single atoms into a collection, or system, of atoms. Each atom that experiences the field can excite to the same energy level or to a slightly different energy level. Thus, by looking at how electrons excite in a single atom we can generalize to a larger system of atoms. Thus we can generalize from the nuclear scale to the atomic scale.

The common notation in quantum mechanics to denote states is *bra-ket*, or *Dirac*, notation. The *ket*, $\mid V\rangle$, is just a column vector with the *bra*, $\langle V \mid$, being a row vector that is the complex transpose of the ket in this case since V is used in both the bra and ket. Since kets are vectors, then we can draw from linear algebra and say that kets can be represented by a unique set of n linearly independent vectors in an n-dimensional space called a basis. Thus we can write a mathematical definition for the expansion of a ket where the vectors $\mid i\rangle$ form a basis.

$$\mid V\rangle = \sum_{i=1}^{n} v_i \mid i\rangle$$

This allows us to write states as a sum of linearly independent orthonormal vectors with coefficients that are the normalized components of the vector.

An issue is that we can have a vector space with no preassigned definition of length or direction for the elements. This is remedied by using the dot product. Thus, we define a generalization called the *inner product* between two vectors $\mid V\rangle$ and $\mid W\rangle$. The inner product is denoted by $\langle V \mid W\rangle$. It is generally a complex number dependent on the two vectors. The inner product for vectors must coincide with axioms of the dot product. Thus, the inner product must obey the following axioms.

1. $\langle V \mid W\rangle = \langle W \mid V\rangle^*$ (skew-symmetry)
2. $\langle V \mid V\rangle \geq 0$ and equal to 0 iff $\mid V\rangle = 0$ (positive semidefiniteness)
3. $\langle V \mid (a \mid W\rangle + b \mid Z\rangle) \equiv \langle V \mid aW + bZ\rangle = a\langle V \mid W\rangle + b\langle V \mid Z\rangle$ (linearity in ket)

From these axioms three concepts can be defined. Two vectors are *orthogonal* or perpendicular if their inner product vanishes. Length or magnitude of a vector is referred to as the *norm* of the vector and is defined as $\sqrt{\langle V \mid V\rangle} \equiv |V|$. A *normalized* vector has unit norm. A set of basis vectors are all of unit norm that are pairwise orthogonal is called an *orthonormal basis*.

Now we can define a formula for evaluating the inner product in terms of its components. Given two kets $\mid V\rangle$ and $\mid W\rangle$ such that

$$\mid V\rangle = \sum_i v_i \mid i\rangle$$

and

$$\mid W\rangle = \sum_j w_j \mid j\rangle$$

and then following the axioms obeyed by the inner product we can obtain an initial formula.

$$\langle V \mid W\rangle = \sum_i \sum_j v_i^* w_j \langle i \mid j\rangle \tag{21}$$

Though now we are left to find the inner product between basis vectors $\langle i \mid j\rangle$. To get around this we just invoke the *Gram-Schmidt* theorem that states if given a linearly independent basis we can form linear combinations of the basis vectors to obtain an orthonormal basis. Assuming that the current basis is orthonormal, then

$$\langle i \mid j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \equiv \delta_{ij},$$

where δ_{ij} is the *Kronecker delta symbol* (Shankar 1994). Feeding this relation into equation (1), the double sum collapses into a single sum due to the Kronecker delta resulting in a simplified form of the inner product.

$$\langle V \mid W\rangle = \sum_i v_i^* w_i$$

It is now clear that the inner product is just the matrix product of the column vector $\mid W\rangle$ and the row vector $\langle V \mid$.

$$\langle V \mid W\rangle = [v_1^*, v_2^*, \ldots, v_n^*] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

Returning to the Gram-Schmidt Theorem, we can now talk about a procedure for converting a basis into an orthonormal one. The easiest way to do this is through a simple two-dimensional example. Let us take two basis vectors in a plane, $\mid A\rangle$ and $\mid B\rangle$. To get an orthonormal basis out of these vectors simply do the following:

1. Rescale the first vector, $\mid A\rangle$, by its own length, $|A| = \sqrt{\langle A \mid A\rangle}$, turning it into a unit vector and the first basis vector.

$$\mid 1\rangle = \frac{\mid A\rangle}{|A|}$$

2. Subtract from the second vector the projection along the first vector leaving behind the part of the second vector that is perpendicular to the first vector.
$$\mid 2'\rangle = \mid B\rangle - \mid 1\rangle\langle 1 \mid B\rangle$$

3. Rescale $\mid 2'\rangle$ by its norm to get a vector orthogonal to the first and of unit length.
$$\mid 2\rangle = \frac{\mid B\rangle - \mid 1\rangle\langle 1 \mid B\rangle}{\sqrt{\langle 2' \mid 2'\rangle}}$$

This procedure can be continued for more than two dimensions which leads us to the understanding that the dimensionality of a space equals the maximum number of mutually orthogonal (i.e. linearly independent) vectors in it.

This leads us to two important and powerful theorems, *the Schwartz Inequality* and *the Triangle Inequality*. The Schwartz Inequality says that the inner product of two vectors cannot exceed the product of their lengths.

$$|\langle V \mid W\rangle| \leq |V||W|$$

The Triangle Inequality simply states that the length of a sum cannot exceed the sum of the lengths.
$$|V + W| \leq |V| + |W|$$

Proofs of both follow immediately: 1. The Schwartz Inequality: Apply the axiom $\langle Z \mid Z\rangle \geq 0$ to
$$\mid Z\rangle = \mid V\rangle - \frac{\langle W \mid V\rangle}{|W|^2} \mid W\rangle.$$

We get

$$\begin{aligned}\langle Z \mid Z\rangle &= \langle V - \frac{\langle W \mid V\rangle}{|W|^2} W \mid V - \frac{\langle W \mid V\rangle}{|W|^2} W\rangle \\ &= \langle V \mid V\rangle - \frac{\langle W \mid V\rangle\langle V \mid W\rangle}{|W|^2} - \frac{\langle W \mid V\rangle^*\langle W \mid V\rangle}{|W|^2} \\ &\quad + \frac{\langle W \mid V\rangle^*\langle W \mid V\rangle\langle W \mid W\rangle}{|W|^4} \\ &\geq 0.\end{aligned}$$

Using
$$\langle W \mid V\rangle^* = \langle V \mid W\rangle$$

we find
$$\langle V \mid V\rangle \geq \frac{\langle W \mid V\rangle\langle V \mid W\rangle}{|W|^2}.$$

Cross multiplying by $|W|^2$ and then taking the square root of both sides derives the Schwartz Inequality.

$$|\langle W \mid W\rangle||\langle V \mid V\rangle| \geq |\langle W \mid V\rangle\langle V \mid W\rangle|$$

2. The Triangle Inequality: It suffices to show that $|a+b|^2 \leq (|a|+|b|)^2$ since we can just take the square root of both sides to get exactly the triangle inequality.

$$\begin{aligned} |a+b|^2 &= (a+b)(a+b) \\ &= a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 \\ &= (|a| + |b|)^2 \end{aligned}$$

A *linear operator*, Ω, is just a matrix that has instruction for transforming any given vector $\mid V\rangle$ into another vector, $\mid V'\rangle$.

$$\mid V\rangle = \Omega \mid V'\rangle$$

This linear operation is not limited to just kets but can also be applied to bras. To be considered a linear operator the following rules must be obeyed.

$$\Omega\alpha \mid V_i\rangle = \alpha\Omega \mid V_i\rangle$$

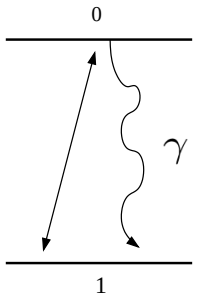
$$\Omega(\alpha \mid V_i\rangle + \beta \mid V_j\rangle) = \alpha\Omega \mid V_i\rangle + \beta\Omega \mid V_j\rangle$$

$$\langle V_i \mid \alpha\Omega = \langle V_i \mid \Omega\alpha$$

$$(\langle V_i \mid \alpha + \langle V_j \mid \beta)\Omega = \alpha\langle V_i \mid \Omega + \beta\langle V_j \mid \Omega$$

Chapter 5

The 2 Level System



The 2 level system consists of a ground state, we will call 1, and an excited state that we will call 0. There is also a natural decay rate from the excited state into the ground state called, γ. Thus the change in the excited state population is

$$\begin{aligned} \dot{n_0} &= In_1 - \gamma n_0 - In_0 \\ &= I(n_1 - n_0) - \gamma n_0 \qquad (22) \end{aligned}$$

where I is the intensity of the light field and the $-In_0$ term is due to stimulated emission. These are conservative so that if you add up the populations of all states you get the same total, namely 1. This is analogous to the trace of the 2×2 matrix that makes up the 2 level systems density matrix.

$$A = \begin{pmatrix} n_{11} & n_{10} \\ n_{01} & n_{00} \end{pmatrix}$$

$$\mathrm{Tr}(A) = 1 = n_1 + n_0$$

$$n_1 = 1 - n_0$$

For convenience we will use $n_{11} = n_1$ and $n_{00} = n_0$ to denote the probabilities that populations of atoms are in the ground and excited states. Thus, equation (22) can be rewritten as

$$\dot{x_0} = I(1 - 2n_0) - \gamma n_0.$$

The steady state is of interest to us. Thus, we set $\dot{n_0} = 0$ and solve for the excited state population density.

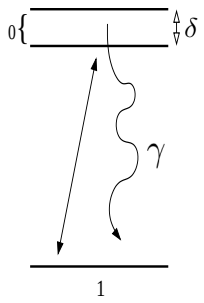
$$\begin{aligned} I(1 - 2n_0) &= \gamma n_0 \\ I &= (\gamma + 2I)n_0 \\ n_0 &= \frac{I}{\gamma + 2I} \\ n_0 &= \frac{1}{\frac{\gamma}{I} + 2} \end{aligned}$$

In this final form we can think of how the population changes at different intensities. Thus, as $I \to 0$ then $n_0 \to 0$. This implies that as the intensity vanishes so does the excited state populations. As $I \to \infty$ then $n_0 \to \frac{1}{2}$. Thus, as intensity increases without bound then $\frac{1}{2}$ of the total population of atoms are found

in the excited state. This is the maximum number of atoms that can occupy the excited state. Decay rates naturally increase for the system to ensure the condition that for the 2 level system only $\frac{1}{2}$ the atoms occupy the excited state.

Chapter 6

Amplitude of the Driven DHO



The beauty of the driven DHO is that it approximates perfectly many situations in nature. The reason that we looked to the driven DHO was the need to describe more accurately the observations of lasing from a continuous wave light emitting diode. Really it is a physical property that causes this fluctuation in the intensity of the laser beam. Since photons travel as waves, when you take a crystal that has surfaces that are reflective to a certain wave length, then you build up a coherent wave of photons along certain wave lengths that can completely fit within the crystals dimensions. Our intention in finding a way to rescale the intensity was due to one of the main ideas that separates our theory from the many other models that have been developed up to this point. Many other Rubidium 87 modeling attempts have hard to achieve processes like supercooling the Rubiudium and/or using a second probe laser to take measurements. The probe lasers mainly come in two orientations, parallel and perpendicular to the travel of the first beam. The two key features here are that we are operating in a regime of atoms that are at room temperature or greater and that instead of two laser beams we use the physics of one beam to take the place of the probe laser. This allows for more practical applications of the theory towards a working product.

Ultimately, we looked at the driven DHO to scale the intensity giving a range of intensities that affect excitations in the atoms differently. We called this the ***two photon detuning***, δ. To find the factor that rescales the intesity we just have to solve for the magnitude of the amplitude, $|A|$, in the driven DHO. The procedure for which we completed this task follows.

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{2m}e^{i\omega t}$$
$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{2m}e^{i\omega t} \qquad (23)$$

Assuming a solution for x to be an amplitude, A, multiplied by an exponential we then take first and second derivatives and substitute back into equation (22) to get the following:

$$x = Ae^{\alpha t}$$
$$\dot{x} = A\alpha e^{\alpha t}$$
$$\ddot{x} = A\alpha^2 e^{\alpha t}$$

$$A\alpha^2 e^{\alpha t} + 2\gamma A\alpha e^{\alpha t} + \omega_0^2 A e^{\alpha t} = \frac{F_0}{2m}e^{i\omega t}.$$

Now noticing that $\alpha = i\omega$, we can cancel all the exponents from the equation and solve for A.

$$\begin{aligned} A\alpha^2 + 2\gamma A\alpha + \omega_0^2 A &= \frac{F_0}{2m} \\ A(\alpha^2 + 2\gamma\alpha + \omega_0^2) &= \frac{F_0}{2m} \\ A &= \frac{F_0}{2m(\alpha^2 + 2\gamma\alpha + \omega_0^2)} \\ &= \frac{F_0}{2m(\omega_0^2 - \omega^2 + i2\gamma\omega)} \end{aligned}$$

$$|A| = \frac{F_0}{2m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \tag{24}$$

Thus, if $\omega = \omega_0$ then

$$\begin{aligned} |A| &= \frac{F_0}{2m\sqrt{4\gamma^2\omega_0^2}} \\ &= \frac{F_0}{4m\gamma\omega_0}. \end{aligned} \tag{25}$$

This situation is called resonance. Dividing the off resonance equation, equation (24), by the on resonance equation, equation (25), we get the factor that scales the intensity and thus detunes the excited state population.

$$\frac{\dfrac{F_0}{2m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}}{\dfrac{F_0}{4m\gamma\omega_0}} = \frac{2\gamma\omega_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

Thus the scaling factor is

$$I' = I\frac{2\gamma\omega_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}.$$

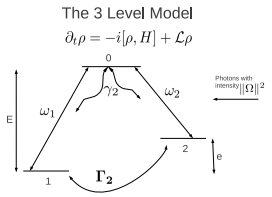
Taking the equation from the previous section for population density in the excited state we just substitute I' in for I.

$$\begin{aligned} n_0 &= \frac{1}{2 + \dfrac{\gamma}{I'}} \\ &= \frac{1}{2 + \dfrac{\gamma\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}{2\omega_0\gamma\ I}} \\ &= \frac{1}{2 + \dfrac{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}{2\omega_0 I}} \end{aligned}$$

This allows the movement of more or less atoms into the excited state based on the fluctuations in the intensity of the beam. This fluctuation is commonly refered to as the *noise* in the laser beam.

Chapter 7

The 3 Level System



The 3 level system is a natural place to start taking a serious look at quantum mechanical processes occurring when a light field interacts with a system of rubidium atoms. This is because in the true scheme of the rubidium system there exists this 3 level lambda system embedded in a similar 5 level system that is described in the next section. The simplicity of the 3 level system gives us insight on how to expect the 5 level system to respond. Also, while coding the theory in C for the 3 level system we essentially developed the part of the final model that is the embedded lambda system inside the double lambda (5 level) system.

The above diagram needs a brief explanation. The term Γ_2 is the natural rate at which the ground states tend toward equilibrium. The term γ_2 is the natural excited state decay rate. The terms ω_1 and ω_2 are the frequencies of the atoms as they are excited from the ground state due to the intensity of the light field. Since the light field fluctuates, then so do the frequencies of the atoms as they are excited. This is known as noise and the use of noise as a tool to probe a system is called ***noise spectroscopy***.

There is a linear partial differential equation that we use to model the system.

$$\partial_t \rho = -i[\rho, H] + \mathcal{L}\rho \tag{25}$$

The first part is $-i[\rho, H]$ which is the commutation relation.

$$[\rho, H] = \rho H - H\rho$$

The greek letter ρ represents a density matrix with on diagonal elements being the populations of atoms in the 3 available states.

$$\begin{bmatrix} \rho_{11} & \rho_{12} & \rho_{10} \\ \rho_{21} & \rho_{22} & \rho_{20} \\ \rho_{01} & \rho_{02} & \rho_{00} \end{bmatrix}$$

The H represents a matrix called the *Hamiltonian*. The Hamiltonian represents the various energy configurations of the atoms (i.e. the atom's energy space).

$$H = \begin{bmatrix} 0 & 0 & \Omega_1 e^{-i\omega_1 t} \\ 0 & e & \Omega_2 e^{-i\omega_2 t} \\ \Omega_1^* e^{i\omega_1 t} & \Omega_2^* e^{i\omega_2 t} & E \end{bmatrix}$$

The Ω_1 and Ω_2 are both rabi frequencies. A rabi frequency is the frequency of periodic exchange of energy and of the oscillation of the occupation probabilities.

To find the intensity for each one just find the magnitude of each by multiplying by the complex conjugate then squaring the magnitude.

$$|\Omega_1|^2 = \Omega_1^* \Omega_1$$

$$|\Omega_2|^2 = \Omega_2^* \Omega_2$$

The ω_1 and ω_2 represent the photon's frequency from each respective rabi frequency. The two values along the diagonal, e and E, represent the energy levels of the ground states. The value E is the largest energy gap from the ground state closest to the nucleus to the excited state. And e is the difference in energy levels in the ground state. Ground state splittings are called Zeeman effect and are attributed to the interaction between a magnetic field and the magnetic dipole moment associated with the orbital angular momentum of the atom.

Finally, we come to the Louisvillian, $\mathcal{L}$. The Louisvillian is another matrix that adjusts the dynamic equations due to the natural decay rates of the states of the atoms.

$$\mathcal{L}\rho = \begin{bmatrix} \frac{\gamma_1}{2}\rho_{00} - \Gamma_1\rho_{11} + \Gamma_1\rho_{22} & -\Gamma_2\rho_{12} & -\gamma_2\rho_{10} \\ -\Gamma_2\rho_{21} & \frac{\gamma_1}{2}\rho_{00} + \Gamma_1\rho_{11} - \Gamma_1\rho_{22} & -\gamma_2\rho_{20} \\ -\gamma_2\rho_{01} & -\gamma_2\rho_{02} & -\gamma_1\rho_{00} \end{bmatrix}$$

Performing the necessary matrix operations from equation (25) we get a single 3×3 matrix representing the dynamic equations. This matrix is too large to present in full so the elements of the matrix are presented separately with the knowledge that the upper triangle of the matrix is just the complex conjugate of the lower triangle of the matrix. We are also now working in the $\Omega_1 = \Omega_2 = \Omega$ regime.

$$\partial_t\rho_{11} = \frac{\gamma_1}{2}\rho_{00} - \Gamma_1\rho_{11} + \Gamma_1\rho_{22} + i(\rho_{01}\Omega e^{-i\omega_1 t} - \rho_{10}\Omega^* e^{i\omega_1 t})$$

$$\partial_t\rho_{22} = \frac{\gamma_1}{2}\rho_{00} + \Gamma_1\rho_{11} - \Gamma_1\rho_{22} + i(\rho_{02}\Omega e^{-i\omega_2 t} - \rho_{20}\Omega^* e^{i\omega_2 t})$$

$$\partial_t\rho_{00} = -\gamma_1\rho_{00} i(\rho_{10}\Omega^* e^{i\omega_1 t} - \rho_{01}\Omega e^{-i\omega_1 t}) + i(\rho_{20}\Omega^* e^{i\omega_2 t} - \rho_{02}\Omega e^{-i\omega_2 t})$$

$$\partial_t\rho_{01} = i(\rho_{01}E + \rho_{21}\Omega^* e^{i\omega_2 t} - \Omega^* e^{i\omega_1 t}(\rho_{00} - \rho_{11})) - \gamma_2\rho_{01}$$

$$\partial_t\rho_{02} = i(\rho_{02}(E - e) + \rho_{12}\Omega^* e^{i\omega_1 t} - \Omega^* e^{i\omega_2 t}(\rho_{00} - \rho_{22})) - \gamma_2\rho_{02}$$

$$\partial_t\rho_{12} = i(\rho_{21}e + \rho_{01}\Omega e^{-i\omega_2 t} - \rho_{20}\Omega^* e^{i\omega_1 t}) - \Gamma_2\rho_{21}$$

The form of the above equations is not adequate since we are in a frame of reference that hides the one and two photon detunings. Thus, a transformation of the current frame is needed to acquire a different reference frame (i.e. inside the system of atoms). Using Dirac notation we calculated the transformation of the density matrix elements.

$$\begin{bmatrix} 0' \\ 1' \\ 2' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\omega_1 t} & 0 \\ 0 & 0 & e^{i\omega_2 t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$
\begin{aligned}
|\ 0'\rangle &= |\ 0\rangle \\
\langle 0'\ | &= \langle 0\ | \\
|\ 1\rangle &= e^{-i\omega_1 t}\ |\ 1'\rangle \\
\langle 1\ | &= \langle 1'\ |\ e^{i\omega_1 t} \\
|\ 2\rangle &= e^{-i\omega_2 t}\ |\ 2'\rangle \\
\langle 2\ | &= \langle 2'\ |\ e^{i\omega_2 t} \\
\rho_{00} &= \rho_{0'0'} \\
\rho_{01} &= |\ 0\rangle\langle 1\ | = |\ 0'\rangle\langle 1'\ |\ e^{i\omega_1 t} = e^{i\omega_1 t}\rho_{0'1'} \\
\rho_{10} &= e^{-i\omega_1 t}\rho_{1'0'} \\
\rho_{02} &= mid0\rangle\langle 2\ | = |\ 0'\rangle\langle 2'\ |\ e^{i\omega_2 t} = e^{i\omega_2 t}\rho_{0'2'} \\
\rho_{20} &= e^{-i\omega_2 t}\rho_{2'0'} \\
\rho_{12} &= |\ 1\rangle\langle 2\ | = e^{-i\omega_1 t}\ |\ 1'\rangle\langle 2'\ |\ e^{i\omega_2 t} = e^{i(\omega_2-\omega_1)t}\rho_{1'2'} \\
\rho_{21} &= e^{-i(\omega_2-\omega_1)t}\rho_{2'1'}
\end{aligned}
$$

Even though we have transformed into the primed states, from here on out we will assume when writing ρ_{10} that we mean $\rho_{1'0'}$ and so on for all the elements in the density matrix. Again we have a conservative system, thus the trace of the density matrix equals 1. We also introduce a variable, d, that is the difference in the ground state populations.

$$
\begin{aligned}
d &= \rho_{11} - \rho_{22} \\
1 &= \rho_{11} + \rho_{22} + \rho_{00} \\
1 - \rho_{11} + \rho_{22} &= 2\rho_{22} + \rho_{00} \\
1 - d - \rho_{00} &= 2\rho_{22} \\
\rho_{22} &= \frac{1 - d - \rho_{00}}{2} \qquad (26) \\
\rho_{11} &= \frac{1 + d - \rho_{00}}{2} \qquad (27)
\end{aligned}
$$

This transformation uncovers the one and two photon detunings when you substitute into the the dynamic equations.

$$
\begin{aligned}
\omega_2 - \omega_1 - e &= \Delta = \delta_1 - \delta_2 \\
E - \omega_1 &= \delta_1 \\
E - e + \omega_2 &= \delta_2
\end{aligned}
$$

After substituting and performing all the necessary operations, we then solve for the partial derivatives of the density matrix elements. What we mean by this is that the partial derivatives of the density matrix elements in the new frame must be taken. This in most cases introduces an exponent and a variable. Then

we can see that every element in the equation now has the same exponent that can be canceled out. We are interested only in the steady state solutions, so we just set those equations equal to 0 and solve for the density matrix element that was the same element at that of the partial derivative we just set equal to zero. We will now show the solutions to the on diagonal elements of the density matrix followed by how we solved for the first off diagonal matrix element. The final solutions for the rest of the off diagonal density matrix elements will round up the list knowing that the lower triangle is the complex conjugate of the upper triangle.

$$\gamma_1\rho_{00} = -i(\rho_{01}\Omega - \rho_{10}\Omega^*) + i(\rho_{02}\Omega - \rho_{20}\Omega^*) \tag{28}$$

We now know from equations (26) and (27) what the solutions for ρ_{11} and ρ_{22} are in terms of ρ_{00} and d. Thus, we need to do some further work on the equation for ρ_{00} by finding $\Omega\rho_{01}$, $\Omega^*\rho_{10}$, $\Omega\rho_{02}$, and $\Omega^*\rho_{20}$. Going from the old frame into the new frame of reference,

$$\begin{aligned}
\partial_t\rho_{01} &= \partial_t(e^{i\omega_1 t}\rho_{01}) \\
&= i\omega_1 e^{i\omega_1 t}\rho_{01} + e^{i\omega_1 t}\partial_t\rho_{01} \\
i\omega_1 e^{i\omega_1 t}\rho_{01} + e^{i\omega_1 t}\partial_t\rho_{01} &= -\gamma_2\rho_{01}e^{i\omega_1 t} + iE\rho_{01}e^{i\omega_1 t} + i\rho_{21}\Omega^* e^{i\omega_1 t} \\
&\quad - i\Omega^* e^{i\omega_1 t}(\rho_{00} - \rho_{11}) \\
&= -(\gamma_2 - i\delta_1)\rho_{01} + i\Omega^*(\rho_{21} + \frac{1}{2} + \frac{d}{2} - \frac{3}{2}\rho_{00})
\end{aligned}$$

$$\begin{aligned}
\rho_{01} &= \frac{i\Omega^*(\rho_{21} + \frac{1}{2} + \frac{d}{2} - \frac{3}{2}\rho_{00})}{(\gamma_2 - i\delta_1)} \\
\rho_{10} &= \frac{-i\Omega(\rho_{21} + \frac{1}{2} + \frac{d}{2} - \frac{3}{2}\rho_{00})}{(\gamma_2 + i\delta_1)} \\
\rho_{02} &= \frac{i\Omega^*(\rho_{21} + \frac{1}{2} - \frac{d}{2} - \frac{3}{2}\rho_{00})}{(\gamma_2 - i\delta_2)} \\
\rho_{20} &= \frac{-i\Omega(\rho_{21} + \frac{1}{2} - \frac{d}{2} - \frac{3}{2}\rho_{00})}{(\gamma_2 + i\delta_2)}.
\end{aligned}$$

Then, since we are in the steady state, $\partial_t\rho_{11} = 0 = \partial_t\rho_{22}$ and we can set the equations equal to each other and solve for d, the difference in ground state populations.

$$\frac{\gamma_1}{2}\rho_{00} - \Gamma_1 d + i\rho_{01}\Omega - i\rho_{10}\Omega^* = \frac{\gamma_1}{2}\rho_{00} + \Gamma_1 d + i\rho_{02}\Omega - i\rho_{20}\Omega^*$$

$$\begin{aligned}
2\Gamma_1 d = &-\frac{|\Omega|^2}{\gamma_2^2 + \delta_1^2}\left[\left(\rho_{21} + \frac{1}{2} + \frac{d}{2} - \frac{3}{2}\rho_{00}\right)(\gamma_2 + i\delta_1)\right] \\
&-\frac{|\Omega|^2}{\gamma_2^2 + \delta_1^2}\left[\left(\rho_{12} + \frac{1}{2} + \frac{d}{2} - \frac{3}{2}\rho_{00}\right)(\gamma_2 - i\delta_1)\right] \\
&+\frac{|\Omega|^2}{\gamma_2^2 + \delta_2^2}\left[\left(\rho_{12} + \frac{1}{2} - \frac{d}{2} - \frac{3}{2}\rho_{00}\right)(\gamma_2 + i\delta_2)\right]
\end{aligned}$$

$$+\frac{|\Omega|^2}{\gamma_2^2+\delta_2^2}\left[\left(\rho_{21}+\frac{1}{2}-\frac{d}{2}-\frac{3}{2}\rho_{00}\right)(\gamma_2-i\delta_2)\right]$$

Now since we are using one light field and then converting into left and right circularly polarized, the one photon detunings are equal, $\delta_1 = \delta_2$, greatly simplifying the equation.

$$\begin{aligned} d &= -\frac{|\Omega|^2}{\Gamma_1(\gamma_2^2+\delta^2)}\left(\gamma_2 d + i\delta(\rho_{21}-\rho_{12})\right) \\ &= -\frac{i\delta|\Omega|^2(\rho_{21}-\rho_{21})}{\Gamma_1^2(\gamma_2^2+\delta^2)^2+\Gamma_1\gamma_2|\Omega|^2(\gamma_2^2+\delta^2)} \end{aligned}$$

Now we have to find the equations for ρ_{21} and ρ_{12} so that we can compute $(\rho_{21}-\rho_{12})$. After several pages of equations where we set the dynamic equations for ρ_{21} and ρ_{12} equal to zero and solved for as few variable as possible we reached equations of two independent variables.

$$\rho_{12} = -\frac{\gamma_2|\Omega|^2(|\tilde{\Gamma_2}+i\Delta)}{(\gamma_2^2+\delta^2)(\tilde{\Gamma_2}+\Delta^2)}\left(1-3\rho_{00}-\frac{id\delta}{\gamma_2}\right)$$

$$\rho_{21} = -\frac{\gamma_2|\Omega|^2(|\tilde{\Gamma_2}-i\Delta)}{(\gamma_2^2+\delta^2)(\tilde{\Gamma_2}+\Delta^2)}\left(1-3\rho_{00}+\frac{id\delta}{\gamma_2}\right)$$

This is the beginning of some substitutions to simplify things. The new value of $\tilde{\Gamma_2}$ is equal to Γ_2 plus another value dependent on the intensity.

$$\tilde{\Gamma_2} = \Gamma_2 + \frac{2\gamma_2|\Omega|^2}{\gamma_2^2+\delta^2}$$

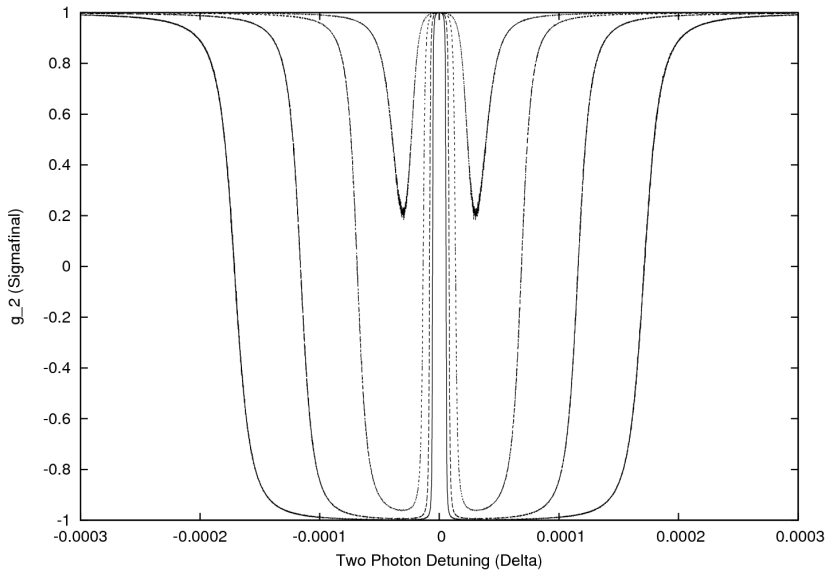
Now the value of $(\rho_{21}-\rho_{12})$ can be computed as well as $(\rho_{21}+\rho_{12})$ for finding ρ_{00}.

$$\rho_{21}-\rho_{12} = -\frac{2i\gamma_2\Delta|\Omega|^2}{(\gamma_2^2+\delta^2)(\tilde{\Gamma_2}^2+\Delta^2)}\left(\frac{d\delta\tilde{\Gamma_2}}{\gamma_2\Delta}-1+3\rho_{00}\right)$$

$$d = -\frac{2\delta\Delta\gamma_2|\Omega|^4}{\Gamma_1(\gamma_2^2+\delta^2)^2(\tilde{\Gamma_2}^2+\Delta^2)+\gamma_2|\Omega|^2(\gamma_2^2+\delta^2)(\tilde{\Gamma_2}^2+\Delta^2)}$$

$$\rho_{21}+\rho_{12} = -\frac{\gamma_2|\Omega|^2}{(\gamma_2^2+\delta^2)(\tilde{\Gamma_2}^2+\Delta^2)}\left[2\tilde{\Gamma_2}-6\tilde{\Gamma_2}\rho_{00}+\frac{2d\delta\Delta}{\gamma_2}\right]$$

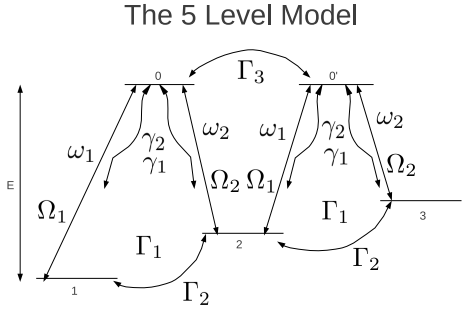
$$\rho_{00} = \frac{2\frac{\gamma_2}{\gamma_1}|\Omega|^2}{(\gamma_2^2+\delta^2)+6\frac{\gamma_2}{\gamma_1}|\Omega|^2}\left[(\rho_{21}+\rho_{12})+1\right]$$



These correlations between states and the fact that rates that connect the excited state to the two ground states are much faster than the rate that connects the two ground states gives rise to the actual physical process that is the main point of interest in our research. To graph this we compared it to experiment and took a statistical average essentially writing code that solved for the elements of the density matrix thousands of times over at different values. This generated one column of data that was how much the two photon detuning is correlated, 1, or anti-correlated, -1, for each random number and another column that was the two photon detuning forming the following graph that shows power broadening of the electromagnetically induced transparency spike that is noticed when the two light fields frequencies become resonant. This shows that at the large two photon detuning limit the system tends towards correlation. But as the two-photon detuning goes to zero the system becomes anti-correlated until you get close to resonance and then there is a sharp spike towards correlation in the system, peaking as the frequencies become resonant.

Chapter 8

The 5 Level System



The 5 level system is essentially two 3 level systems that share a ground state. This adds complexity to the system making it more difficult to obtain solutions as we did for the 3 level system. The same assumptions for the 3 level are made for the 5 level (i.e. the trace of the density matrix equals 1) with the additional assumption that there is very little population in the excited states. We also make the assumption that

$$\rho_{11} = \frac{1}{3} - d$$

and

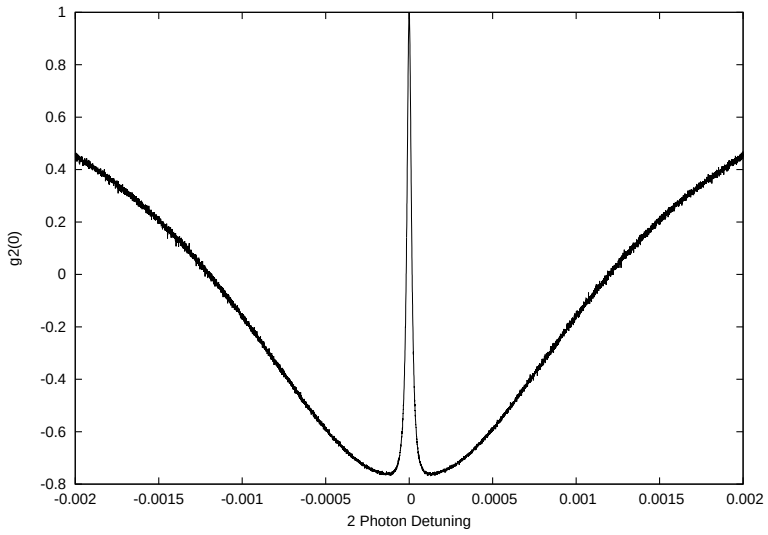
$$\rho_{33} = \frac{1}{3} - b$$

where d is a linear combination of ρ_{21} and ρ_{12} and b is a linear combination of ρ_{32} and ρ_{23}.

The above graph has familiar elements from the 3 level system with a few additonal elements. The term Γ_1 is the rate of population difference between the ground states. The term Γ_2 is the rate of decoherence between the ground states. The term Γ_3 is the rate of decohernece between the excited states. The term γ_1 is the rate at which the excited state populations decay to the ground state. The term γ_2 is the rate at which the excited state coherence decays to zero.

The key difference between the two systems is the extra ground state coherence. This extra ground state tends to mix up the coherences making it easier to be anti-correlated and harder to become correlated. Since the 5 level system has this double lambda structure, a multi-photon coherence arises connecting ground states through the excited states.

The 5 level has a broader EIT spike though the inner spike thins up due to the added movement of atoms from the connected third ground state. This third ground state makes it harder to be correlated at large two photon detuning (Δ) as well as becoming harder to be correlated as the two photon detuning goes to zero.


g2(0)
2 Photon Detuning
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.002
-0.0015
-0.001
-0.0005
0
0.0005
0.001
0.0015
0.002

Part IV

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