An Intriguing Limit with Euler in Mind

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Abstract

The purpose of this paper is to present the solution to the limit,

$$\lim_{n\to\infty} \sqrt[n]{\operatorname{lcm}(1,2,\ldots,n)}$$

including the techniques and results needed to evaluate the limit. First, the technique of partial summation is introduced. Then a few lemmas are proved which lead to the solution.

In solving this, the author will also trace some of the steps which Chebyshev followed in his pursuit of the Prime Number Theorem. This will be in the form of proving some of his results, but not necessarily in the manner in which he did.

Notation, Conventions and Preliminaries

Throughout the paper, the convention that if $\{a_n\}_{n=1}^{\infty}$ is a sequence then we have that

$$\sum_{n<1} a_n = 0$$

will be used to simplify notation. Notice that the Number Theoretic convention of using "log" to represent the natural logarithm will also be adopted. Also, when representing variables and indices, the letter p will be reserved solely for primes. The following definition is necessary,

Definition. Let a function $f : \mathbb{Z}^+ \to \mathbb{C}$, then f is called an **arithmetic function**.

As a result, sequences of complex numbers indexed by positive integers may be considered arithmetic functions.

Now, a few basic tools for approximation and analysis are needed. The first is a notation for approximation, called "Big Oh Notation" which is presented as,

Definition. Let f, g be functions with $g(x) > 0$ for all $x \geq a$. We write

$$f(x) = O(g(x)) \quad \text{(read: } f(x) \text{ is big oh of } g(x))$$

if there is some $M > 0$ such that

$$|f(x)| \leq M g(x) \quad \forall x \geq a.$$

Example 1. $\sin(x) = O(1)$ and $\log x = O(\sqrt{x})$

Another approximation notation which shows asymptotic dominance of one function over another is called "Little Oh Notation".

Definition. Let f, g be functions with $g(x) \neq 0$ for x large then we write,

$$f(x) = o(g(x)) \quad \text{(read: } f(x) \text{ is little oh of } g(x))$$

if we have that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

Example 2. $x = o(x^2)$ and $\log x = o(x)$.

Finally, the notion of asymptotic equality of functions is needed.

Definition. Let f, g be functions. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

we say $f(x)$ **is asymptotic to** $g(x)$ and write

$$f(x) \sim g(x) \quad \text{as } x \to \infty.$$

This is quite a strong condition and is sometimes referred to as multiplicative asymptotic equality. Let us work to restate this property using little oh notation.

Lemma 1. *If f, g, h are functions with $f(x) = o(g(x))$, then the following are true:*

(i). $h(x)f(x) = o(h(x)g(x))$

(ii). $\dfrac{f(x)}{h(x)} = o\left(\dfrac{g(x)}{h(x)}\right)$ *if* $h(x) \neq 0$.

Proof. We know that,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

from the definition. Considering the corresponding quotient for (i) we have that

$$\lim_{x\to\infty} \frac{h(x)f(x)}{h(x)g(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

proving (i) Similarly,

$$\lim_{x\to\infty} \frac{f(x)/h(x)}{g(x)/h(x)} = \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

which proves the lemma. □

Corollary $(*)$**.** *Given two functions f, g, with $g(x) \neq 0$ for x large, then $f(x) \sim g(x)$ if and only if $f(x) = g(x) + o(g(x))$.*

Proof. Suppose $f(x) \sim g(x)$. Then,

$$\frac{f(x)}{g(x)} = 1 + o(1)$$

so that $f(x) = g(x) + o(g(x))$ by lemma 1. Conversely if $f(x) = g(x) + o(g(x))$ then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} 1 + o(1) = 1. \qquad \square$$

Partial Summation(Abel Summation)

Theorem 1 provides a technique for evaluating sums of a certain form which proves to be an important tool in mathematics.

Theorem 1 (Partial Summation)**.** *Let $x, y \in \mathbb{R}$ and set $k = \lfloor x \rfloor$ and $m = \lfloor y \rfloor$. Suppose that $\{a_n\}_{n=1}^{\infty}$ is an arithmetic function with*

$$A(x) = \sum_{n \leq x} a_n.$$

Then,

$$\sum_{y<n\leq x} a_n f(n) = \sum_{n=m+1}^{k-1} A(n)(f(n) - f(n+1)) + A(k)f(k) - A(m)f(m+1). \qquad (\dagger)$$

Furthermore, if we have that $f \in C^1(y, x), \quad 0 < y < x$ then,

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt. \qquad (\ddagger)$$

Proof. Examining the left-hand side we have,

$$\begin{aligned}
\sum_{n\leq x} a_n f(n) &= \sum_{n=m+1}^{k} a_n f(n) \\
&= \sum_{n=m+1}^{k} \left(A(n) - A(n-1)\right) f(n) \\
&= \sum_{n=m+1}^{k} A(n)f(n) - \sum_{n=m+1}^{k} A(n-1)f(n) \\
&= \sum_{n=m+1}^{k} A(n)f(n) - \sum_{n=m}^{k-1} A(n)f(n+1) \\
&= \sum_{n=m+1}^{k-1} A(n)(f(n) - f(n+1)) + A(k)f(k) - A(m)f(m+1)
\end{aligned}$$

which proves (†). Now we suppose that $f \in C^1(y, x)$ then,

$$\begin{aligned}
\sum_{n\le x} a_n f(n) &= \sum_{n=m+1}^{k-1} A(n)(f(n) - f(n+1)) + A(k)f(k) - A(m)f(m+1) \\
&= -\sum_{n=m+1}^{k-1} A(n)\int_n^{n+1} f'(t)dt + A(k)f(k) - A(m)f(m+1) \\
&= -\sum_{n=m+1}^{k-1} \int_n^{n+1} A(t)f'(t)dt + A(k)f(k) - A(m)f(m+1)
\end{aligned}$$

since A(t) is a step function. This is equivalent to,

$$\begin{aligned}
&-\int_{m+1}^{k} A(t)f'(t)dt + A(x)f(x) - \int_k^x A(t)f'(t)dt - A(y)f(y) - \int_y^{m+1} A(t)f(t)dt \\
&= A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt
\end{aligned}$$

proving (‡). □

This theorem is due to Abel who first proved the result. In light of this, it is sometimes called Abel Summation. An immediate corollary is

Corollary. *Assuming the setup to theorem 1 and taking $y < 1$ we have*

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.$$

This form will be used throughout the remainder of this paper. A few examples of the power of this technique follow.

Example 3. Show that

$$\sum_{n\le x} \log n = x\log x - x + O(\log x).$$

Solution. With an application of the corollary with $f(n) = \log n$ and $a_n = 1$ we see that,

$$\begin{aligned}\sum_{n\le x} \log n &= \log x \sum_{n\le x} 1 - \int_1^x \frac{1}{t} \sum_{n\le t} 1 dt \\ &= \lfloor x \rfloor \log x - \int_1^x \lfloor t \rfloor \frac{1}{t} dt \\ &= x \log x - \{x\} \log x - \int_1^x 1 dt + \int_1^x \frac{\{t\}}{t} dt \\ &= x \log x - x + O(\log x)\end{aligned}$$

where $\lfloor x \rfloor$ is the greatest integer function and $\{x\}$ denotes the fractional part of x. □

After seeing the examples, let us now work to formulate the solution to the limit. We will need to define a particular arithmetic function known as the Von Mangoldt Function.

Definition. Let $n \geq 1$ be an integer then define,

$$\Lambda(n) = \begin{cases} log\ p & \text{if } n = p^m \text{ for some integer } m \\ 0 & \text{otherwise} \end{cases}$$

This function is called the **Von Mangoldt Function**.

The next lemma will make use of this function and will put us close to a solution of the limit.

Lemma 2. *Define*

$$\psi(x) = \sum_{p^\alpha \le x} log\ p = \sum_{n \le x} \Lambda(n),$$

where p is a prime and Λ is the previously defined Von Mangoldt function. Then, we have

$$\text{lcm}(1, 2, \ldots, n) = e^{\psi(n)}.$$

Proof. First, notice that we can write,

$$\text{lcm}(1, 2, \ldots, n) = \prod_{p \le n} p^{\eta_p}$$

where η_p is the largest integer such that $p^{\eta_p} \le n$. Notice that,

$$\eta_p = \sum_{p^\alpha \le n} 1$$

Then we can also say that,

$$\text{lcm}(1, 2, \dots, n) = \prod_{p \le n} e^{(log\ p)\cdot \eta_p} = e^{\sum_{p \le n} (log\ p)\cdot \eta_p} = e^{\psi(n)}.$$

Which completes the proof. □

Now, reexamining the limit in light of the previous lemma we can see that,

$$\lim_{n\to\infty} \sqrt[n]{\text{lcm}(1, 2, \dots, n)} = \lim_{n\to\infty} \sqrt[n]{e^{\psi(n)}} = \lim_{n\to\infty} e^{\psi(n)/n}$$

It should be noted that ψ is one of the Chebyshev functions which he first introduced in 1848. It has a more restricted counterpart known as ϑ.

Definition. Let $n \in \mathbb{Z}$ then define ϑ as,

$$\vartheta(n) = \sum_{p \le n} \log p$$

where p is prime.

Notice that ϑ is defined over primes while ψ is defined over powers of primes. They are equivalent otherwise. Note that it is possible to write ψ in terms of ϑ as:

$$\psi(x) = \sum_{p^\alpha \le x} \log p = \sum_{\alpha=1}^{\infty} \sum_{p^\alpha \le x} \log p = \sum_{\alpha=1}^{\infty} \sum_{p \le x^{1/\alpha}} \log p.$$

Though the outer sum over α seems infinite, it is actually finite. Notice that the inner sum is empty if, $x^{1/\alpha} < 2$ since it is over the primes. That is if $(1/\alpha)\log x < \log 2$. So, we can write that,

$$\psi(x) = \sum_{\alpha \le \frac{\log x}{\log 2}} \sum_{p \le x^{1/\alpha}} \log p = \vartheta(x^{1/\alpha}). \tag{1}$$

It will be useful to consider ϑ and this relationship, (1) to prove a few results about ψ. First, we examine this relationship in order to write ψ in terms of ϑ.

Lemma 3. *We have that,*

$$\psi(x) = \vartheta(x) + O(x^{1/2} \log^2 x).$$

Proof. From (1) we consider $\vartheta(x^{1/\alpha})$.

$$\psi(x) = \vartheta(x^{1/\alpha}) = \vartheta(x) + \vartheta(x^{1/m}), \quad m \geq 2.$$

Now, let $m \geq 2$ and consider,

$$\begin{aligned}
\vartheta(x^{1/m}) &= \sum_{2 \leq m \leq \frac{\log x}{\log 2}} \sum_{p \leq x^{1/m}} \log p \\
&\leq \sum_{2 \leq m \leq \frac{\log x}{\log 2}} \sum_{n \leq x^{1/2}} \log x \\
&= \sum_{2 \leq m \leq \frac{\log x}{\log 2}} x^{1/2} \log x \\
&\leq \frac{x^{1/2} \log^2 x}{\log 2} = O(x^{1/2} \log^2 x)
\end{aligned}$$

so that $\psi(x) = \vartheta(x) + O(x^{1/2} \log^2 x)$. □

Now, we present a result which links the asymptotic behavior of the two Chebyshev functions.

Corollary. *$\psi(x)/x$ tends to a limit, λ as x tends to infinity if and only if $\vartheta(x)/x$ tends to λ as x approaches infinity.*

Proof. From lemma 3 we first work with the big oh term. It will be shown that $O(x^{1/2} \log^2 x) = o(x)$. So for some constant, $M > 0$ and upon dividing by x consider,

$$\begin{aligned}
&\lim_{x \to \infty} \frac{M \log^2 x}{x^{1/2}} \\
&\stackrel{L'H}{=} \lim_{x \to \infty} \frac{4M \log x}{x^{1/2}} \\
&\stackrel{L'H}{=} \lim_{x \to \infty} \frac{8M}{x^{1/2}} = 0.
\end{aligned}$$

Therefore $O(x^{1/2} \log^2 x) = o(x)$ and it follows

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = \lim_{x \to \infty} \frac{\vartheta(x)}{x} + o(1) = \lim_{x \to \infty} \frac{\vartheta(x)}{x}.$$

showing that the limits must be equal. □

Theorem 2. *The following are equivalent:*

$$\lim_{x\to\infty} \frac{\vartheta(x)}{x} = 1 \tag{2}$$

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = 1 \tag{3}$$

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1. \tag{4}$$

Proof. To prove this we will show that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). The first implication is trivial from the preceeding corollary. Next, suppose that (3) is true. Restating this using Corollary ($*$) we have that $\psi(x) = x + o(x)$. Now, start by applying Abel Summation with $f(n) = 1/\log n$ and $a_n = \Lambda(n)$ so that $A(n) = \psi(n)$, to the summation,

$$\sum_{n\le x} \frac{\Lambda(n)}{\log n} = \frac{\psi(x)}{\log x} + \int_1^x \frac{\psi(t)}{t\log^2 t}dt = \frac{\psi(x)}{\log x} + \int_2^x \frac{\psi(t)}{t\log^2 t}dt \tag{5}$$

since $\psi(1) = 0$. Now, consider the first term on the right hand side of (5),

$$\frac{\psi(x)}{\log x} = \frac{\psi(x)}{x} \cdot \frac{x}{\log x} = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right). \tag{6}$$

We now work with the integral in (5),

$$\int_2^x \frac{\psi(t)}{t\log^2 t}dt < \int_2^x \frac{\psi(t)}{\log^2 t}dt \le \psi(x)\int_2^x \frac{dt}{\log^2 t}. \tag{7}$$

From (7) it readily follows that.

$$\int_2^x \frac{\psi(t)}{t\log^2 t}dt = O\left(\int_2^x \frac{dt}{\log^2 t}\right). \tag{8}$$

One more observation must be made so that π will be involved. Consider again the sum in (5),

$$\begin{aligned}\sum_{n\le x} \frac{\Lambda(n)}{\log n} &= \sum_{1\le m\le \lfloor \frac{\log x}{\log p}\rfloor} \sum_{p^m\le x} \frac{1}{m} \\ &= \pi(x) + \sum_{2\le m\le \lfloor \frac{\log x}{\log p}\rfloor} \sum_{p\le x^{1/m}} \frac{1}{m} \\ &= \pi(x) + \sum_{2\le m\le \lfloor \frac{\log x}{\log p}\rfloor} \pi(x^{1/m}) \qquad (9)\end{aligned}$$

We now consider the sum in (9) to give a bound on the original sum.

$$\sum_{2 \leq m \leq \lfloor \frac{\log x}{\log p} \rfloor} \pi(x^{1/m}) \leq \left\lfloor \frac{\log x}{\log p} \right\rfloor \pi(x^{1/2}) \leq x^{1/2} \log x. \tag{10}$$

Upon combining (9) and (10) we have,

$$\sum_{n \leq x} \frac{\Lambda(n)}{\log n} = \pi(x) + O(x^{1/2} \log x). \tag{11}$$

The final approximation deals with the integral on the right hand side of (8),

$$\begin{aligned}
\int_2^x \frac{dt}{\log^2 t} &= \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t} \\
&= O(\sqrt{x}) + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t} \\
&\leq O(\sqrt{x}) + \frac{1}{\log^2 x^{1/2}} \int_{\sqrt{x}}^x dt \\
&= O(\sqrt{x}) + O\left(\frac{x}{\log^2 x}\right) \\
&= O\left(\frac{x}{\log^2 x}\right)
\end{aligned} \tag{12}$$

Finally, we combine and rearrange the above results as follows. From (11),

$$\pi(x) = \sum_{n \leq x} \frac{\Lambda(n)}{\log n} + O(x^{1/2} \log x).$$

Then, applying (6) and (12),

$$\begin{aligned}
\pi(x) &= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) + O\left(\frac{x}{\log^2 x}\right) \\
&= \frac{x}{\log x} + o\left(\frac{x}{\log x}\right)
\end{aligned}$$

Thus we have demonstrated that $\psi(x) = x + o(x)$ implies that $\pi(x) = x/\log x + o(x/\log x)$. We will now use similar methods to demonstrate the last implication.

Assume that $\pi(x) = x/\log x + o(x/\log x)$. Abel summation is employed again. First, define,

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{otherwise} \end{cases}$$

the characteristic prime function. Recalling the definition of ϑ with the aid of Partial summation we may write,

$$\begin{aligned} \vartheta(x) = \sum_{n \le x} a(n) \log n = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt \\ = x + o(x) - \int_2^x \frac{\pi(t)}{t} dt \\ = x + o(x). \end{aligned} \tag{13}$$

This follows since,

$$\begin{aligned} 0 \le \frac{1}{x} \int_2^x \frac{\pi(t)}{t} dt \le \frac{1}{x} \int_2^x \frac{\pi(x)}{x} dt \\ = \frac{\pi(x)}{x} - \frac{2\pi(x)}{x^2} \\ = \frac{x/\log x + o(x/\log x)}{x} - \frac{2(x/\log x + o(x/\log x))}{x^2} \\ = 1/\log x + o(1/\log x) - \frac{2(1/\log x + o(1/\log x))}{x} \end{aligned}$$

approaches 0 as x approaches infinity. Thus, $\int_2^x \pi(t)/t \; dt = o(x)$. Proving, that $\vartheta(x) = x + o(x)$. □

Now, upon realizing that (4) is the prime number theorem, which was first proved in 1896 independently by Hadamard and de la Vallée Poussin, we have from theorem 2 that $\psi(x) = x + o(x)$. So, considering the limit in question,

$$\lim_{n\to\infty} \sqrt[n]{\text{lcm}(1, 2, \ldots, n)} = \lim_{n\to\infty} \sqrt[n]{e^{\psi(n)}} = \lim_{n\to\infty} e^{\psi(n)/n} = e.$$

It should be noted that the actual form of the prime number theorem proved in 1896 was that represented in (2) or (3). However, as these are less well known, the author chose to present the well known form of the prime number theorem.

Bibliography

[1] Tom M. Apostol. *Introduction to analytic number theory.* Springer-Verlag, 1976.

[2] L. E. Dickson. *History of the theory of numbers*, volume I. 1919. Reprint: Chelsea, 1952, 1966.

[3] M. Ram Murty. *Problems in analytic number theory.* Springer-Verlag, 2001.

[4] W. Narkiewicz. *The development of prime number theory.* Springer-Verlag Berlin-Heidelberg, 2000.