3: Thermionic Emission

Purpose

While we think of quantum mechanics being best demonstrated in processes that show discontinuous change, historically quantum mechanics was first revealed in systems where a large number of particles washed out the jumps: blackbody radiation and thermionic emission. In this lab you will investigate these two phenomena in addition to classical space-charge limited electron emission: Child's Law.

Introduction

Metals, as demonstrated by their ability to conduct an electric current, contain mobile electrons. (Most electrons in metals, particularly the "core" electrons closest to the nucleus, *are* tightly bound to individual atoms; it is only the outermost "valence" electrons that are somewhat "free".) These free electrons are generally confined to the bulk of the metal. As you learned in E&M, an electron attempting to leave a conductor experiences a strong force attracting it back towards the conductor due to an image charge:

$$F_x = -\frac{e^2}{4\pi\epsilon_0 (2x)^2} \tag{3.1}$$

where x is the distance the electron is from the interface and e is the absolute value of the charge on an electron. Of course, inside the metal the electric field is zero so an electron there experiences zero (average) force. You can think of these valence electrons as bouncing around inside a box whose "walls" are provided by the image-charge force. (Odd to think: the "walls" are non-material force fields; the "inside" of the box is filled with solid metal.) Since temperature is a measure of random kinetic energy, if we increase the temperature of the metal, the electrons will be moving faster and some will have enough energy to overcome the image-charge force (which after all becomes arbitrarily small at large distances from the interface) and escape. This is electron "evaporation". The higher the temperature the larger the current of escaping electrons. This temperature induced electron flow is called thermionic emission. Starting in 1901, Owen Richardson studied this phenomenon and in 1929 he received the Nobel prize in Physics for his work.

A hot wire will be surrounded by evaporated electrons. An electric force can pull these electrons away from the wire — the larger the electric force, the larger the resulting current of electrons. The precise relationship between the voltage and the resulting current flow

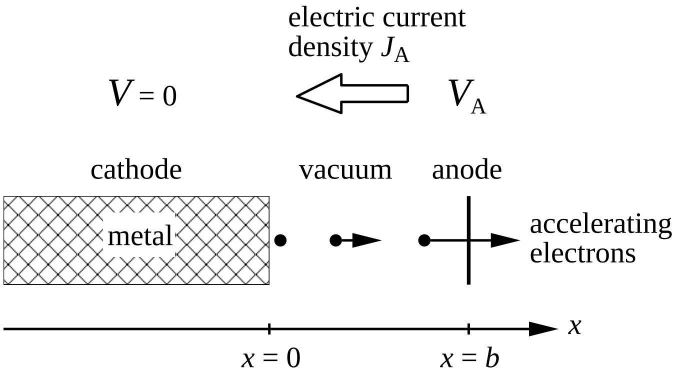

Figure 3.1: A planar cathode and a planar anode are separated by a distance b. A positive potential difference V_A attracts electrons from the cathode to the anode, so the speed of the electrons $v(x)$ increases as they approach the anode. The moving electrons constitute an electric current from anode to cathode. The resulting steady current density is called J_A.

is called Child's law[1] (or the Child-Langmuir law, including Langmuir who independently discovered it while working at G.E.). In this experiment you will measure both Child's Law and the Richardson Effect.

Child's Law

Consider a planar interface between a metal ($x < 0$) and "vacuum" ($x > 0$). Vacuum is in quotes because this region will contain escaped electrons—a 'space charge'—rather than being totally empty[2]. The number of electrons per volume (i.e., the number density) is denoted by n.

In this experiment, the metal will be heated (i.e., its a 'hot cathode' or filament) which will result in a supply of electrons 'evaporated' from the metal into the vacuum. An additional conducting sheet (the anode) is located at $x = b$. A positive potential difference, V_A, between the cathode and the anode plane provides a force pulling these electrons from the vicinity of the cathode towards the anode. The result is a stream of moving electrons (a current); the number density $n(x)$ and speed $v(x)$ of these electrons will depend on location, x, between the plates. The negatively charged electrons moving to the right constitute a steady electric current density to the left, i.e., a steady conventional electric current from the anode to the cathode:

$$J = -en(x)v(x) = -J_A \tag{3.2}$$

Since the electrons leave the metal with (nearly) zero speed at zero potential, we can calculate their speed along the path to the anode using conservation of energy:

$$\frac{1}{2}mv^2 - eV(x) = 0 \tag{3.3}$$

$$v = \sqrt{\frac{2e}{m}\,V(x)} \tag{3.4}$$

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[1] Clement Dexter Child (1868–1933) Born: Madison, Ohio, A.B. Rochester, Ph.D. Cornell

[2] In fact a perfect vacuum is not possible, so the word "vacuum" actually refers simply to a region with relatively few particles per volume

where $V(x)$ is the potential difference ("voltage") at x and m is the mass of an electron. Because the accelerating electrons constitute a steady current (i.e., J_A doesn't depend on position), $n(x)$ must decrease as the electrons speed toward the anode. The varying space charge density affects the electric potential in the "vacuum" according to Poisson's equation[3]:

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho(x)}{\epsilon_0} = \frac{en(x)}{\epsilon_0} \tag{3.5}$$

Putting these pieces together with have the differential equation:

$$\frac{d^2V}{dx^2} = \frac{J_A}{\epsilon_0 v(x)} = \frac{J_A}{\epsilon_0 \sqrt{\frac{2e}{m}\, V(x)}} \tag{3.6}$$

Since the electric field will be zero at the interface, we have a pair of initial conditions:

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} = 0 \tag{3.7}$$

$$\left.V\right|_{x=0} = 0 \tag{3.8}$$

This differential equation looks a bit like Newton's second law:

$$\frac{d^2x}{dt^2} = \frac{1}{m}\, F(x(t)) \tag{3.9}$$

as you can see if in Newton's second law you substitute:

$$\begin{aligned} t &\longrightarrow x \\ x(t) &\longrightarrow V(x) \\ \frac{1}{m}\, F(x(t)) &\longrightarrow \frac{J_A}{\epsilon_0\sqrt{\frac{2e}{m}\, V(x)}} \end{aligned}$$

Recall that force problems are often most simply solved using conservation of energy and that conservation of energy was proved using an integrating factor of dx/dt. If we try the analogous trick on our voltage problem, we'll multiply Poisson's equation by dV/dx:

$$\frac{dV}{dx} \times \frac{d^2V}{dx^2} = \frac{J_A}{\epsilon_0\sqrt{\frac{2e}{m}}}\, V^{-\frac{1}{2}} \times \frac{dV}{dx} \tag{3.10}$$

$$\left(\frac{1}{2}\left[\frac{dV}{dx}\right]^2\right)' = \frac{J_A}{\epsilon_0\sqrt{\frac{2e}{m}}} \left(\frac{V^{\frac{1}{2}}}{\frac{1}{2}}\right)' \tag{3.11}$$

$$\frac{1}{2}\left[\frac{dV}{dx}\right]^2 = \frac{J_A}{\epsilon_0\sqrt{\frac{2e}{m}}}\, \frac{V^{\frac{1}{2}}}{\frac{1}{2}} + \text{constant} \tag{3.12}$$

The initial conditions require the constant to be zero, so

$$\frac{1}{2}\left[\frac{dV}{dx}\right]^2 = \frac{J_A}{\epsilon_0\sqrt{\frac{2e}{m}}}\, \frac{V^{\frac{1}{2}}}{\frac{1}{2}} \tag{3.13}$$

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[3] Poisson's equation is derived in the Appendix to this lab.

or

$$\frac{dV}{dx} = \sqrt{\frac{4J_A}{\epsilon_0\sqrt{\frac{2e}{m}}}}\, V^{\frac{1}{4}} \tag{3.14}$$

This differential equation is separable:

$$\begin{aligned} \frac{dV}{V^{\frac{1}{4}}} &= \sqrt{\frac{4J_A}{\epsilon_0\sqrt{\frac{2e}{m}}}}\, dx & (3.15) \\ \frac{V^{\frac{3}{4}}}{\frac{3}{4}} &= \sqrt{\frac{4J_A}{\epsilon_0\sqrt{\frac{2e}{m}}}}\, x & (3.16) \end{aligned}$$

where again the initial conditions require the constant of integration to be zero. Finally:

$$V(x) = \left[\frac{9J_A}{4\epsilon_0\sqrt{\frac{2e}{m}}}\right]^{\frac{2}{3}} x^{\frac{4}{3}} \tag{3.17}$$

Of course, $V(b)$ is the anode voltage V_A, so we can rearrange this equation to show Child's law:

$$J_A = \left[\frac{4\epsilon_0}{9b^2}\sqrt{\frac{2e}{m}}\right] V_A^{\frac{3}{2}} \tag{3.18}$$

Much of Child's law is just the result of dimensional analysis, i.e., seeking any possible dimensionally correct formula for J_A. Our differential equation just involves the following constants with dimensions (units) as shown:

$$\begin{aligned} b &: L & (3.19) \\ V_A &: \frac{E}{Q} = \frac{ML^2/T^2}{Q} & (3.20) \\ \epsilon_0\sqrt{\frac{2e}{m}} \equiv k &: \frac{Q^2}{EL}\,\frac{Q^{\frac{1}{2}}}{M^{\frac{1}{2}}} = \frac{Q^{\frac{5}{2}}}{M^{\frac{3}{2}}L^3/T^2} & (3.21) \\ J_A &: \frac{Q/T}{L^2} & (3.22) \end{aligned}$$

where the dimensions are: L=length, T=time, M=mass, E=energy, and Q=charge. To make a dimensionally correct formula for J_A, we just eliminate the M dimension which we can only do with the combination:

$$V_A k^{\frac{2}{3}} \quad : \quad \frac{Q^{\frac{2}{3}}}{T^{\frac{2}{3}}} \tag{3.23}$$

We can then get the right units for J_A with:

$$\frac{\left(V_A k^{\frac{2}{3}}\right)^{\frac{3}{2}}}{b^2} = \frac{k}{b^2}\, V_A^{\frac{3}{2}} \quad : \quad \frac{Q/T}{L^2} \tag{3.24}$$

Thus the only possible dimensionally correct formula is

$$J_A \propto \frac{k}{b^2}\, V_A^{\frac{3}{2}} \tag{3.25}$$

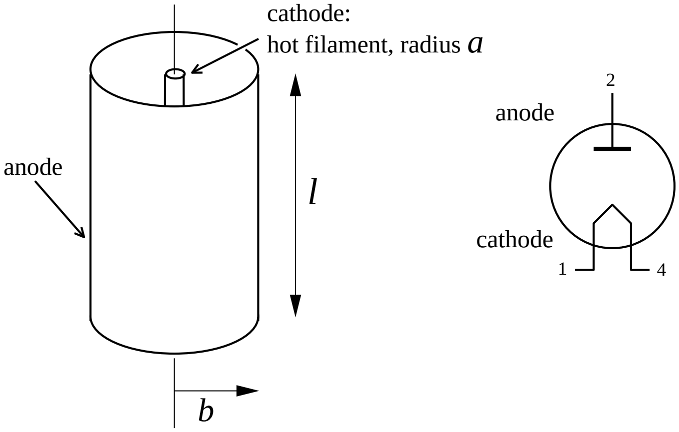

Figure 3.2: Coaxial cylinders: an inner wire (radius a) and outer cylindrical anode (radius b), form a vacuum tube diode. The cathode is heated so electron evaporation is possible, and a potential difference V_A attracts electrons from the cathode to the anode. The speed of the electrons $v(r)$ increases as they approach the anode. The moving electrons constitute a steady electric current from anode to cathode. Since the same current is spread out over larger areas, the current density, J, between the cylinders must be proportional to $1/r$.

The exact proportionality constant, found from the differential equation, is (as usual) is not hugely different from 1.

We have derived Child's law for the case of infinite parallel plates, but you will be testing it in (finite length) coaxial cylinders. The inner wire (radius a) is the cathode; the outer cylinder (radius b) is the anode. Your cylinder with have some length ℓ, but we will below consider infinite length coaxial cylinders. Note that dimensional considerations require that the anode current per length should be given by a formula like:

$$I/\ell \equiv j \propto \frac{k}{b} \, V_A^{\frac{3}{2}} \tag{3.26}$$

although we could have an arbitrary function of the radius ratio: b/a on the right-hand-side.

From Poisson's equation[4] we have:

$$\nabla^2 V = \frac{J}{\epsilon_0 v(r)} = \frac{I}{2\pi r \ell \epsilon_0 v(r)} = \frac{j}{2\pi r \epsilon_0 \sqrt{\frac{2e}{m}}} \, V^{-\frac{1}{2}} \tag{3.27}$$

Using the Laplacian in cylindrical coordinates we find:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} = \frac{j}{2\pi r \epsilon_0 \sqrt{\frac{2e}{m}}} \, V^{-\frac{1}{2}} \tag{3.28}$$

There is no known formula for the solution to this differential equation, but we can make considerable progress by writing the differential equation in terms of dimensionless quanti-

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[4] Poisson's equation is derived in the Appendix to this lab.

ties:

$$\begin{aligned} r/a &= \rho && (3.29) \\ V &= \left[\frac{ja}{2\pi\epsilon_0\sqrt{\frac{2e}{m}}} \right]^{\frac{2}{3}} f(\rho) && (3.30) \end{aligned}$$

yielding:

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\,\frac{\partial f}{\partial \rho} = f''(\rho) + \frac{1}{\rho}\, f'(\rho) = \frac{1}{\rho}\, f^{-\frac{1}{2}} \qquad (3.31)$$

with initial conditions:

$$\begin{aligned} f(1) &= 0 && (3.32) \\ f'(1) &= 0 && (3.33) \end{aligned}$$

We can numerically solve this differential equation using *Mathematica*:

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NDSolve[{f''[p]+f'[p]/p==1/(p Sqrt[f[p]])}, f[1]==0, f'[1]==0, {f},{p,1,200}]
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It's actually not quite that simple. The cathode, at $\rho = 1$, is actually a singular point of the differential equation (i.e., $f''(1) = \infty$). However the situation very near the cathode is well approximated by the planar case, where we've shown:

$$\begin{aligned} V(x) &= \left[\frac{9J_A}{4\epsilon_0\sqrt{\frac{2e}{m}}} \right]^{\frac{2}{3}} x^{\frac{4}{3}} = \left[\frac{9I}{2\pi a\ell 4\epsilon_0\sqrt{\frac{2e}{m}}} \right]^{\frac{2}{3}} (r-a)^{\frac{4}{3}} = \left[\frac{9ja}{2\pi 4\epsilon_0\sqrt{\frac{2e}{m}}} \right]^{\frac{2}{3}} \left(\frac{r-a}{a} \right)^{\frac{4}{3}} \\ &= \left[\frac{9}{4} \right]^{\frac{2}{3}} \left[\frac{ja}{2\pi\epsilon_0\sqrt{\frac{2e}{m}}} \right]^{\frac{2}{3}} \left(\frac{r-a}{a} \right)^{\frac{4}{3}} \qquad (3.34) \end{aligned}$$

So, near the cathode (i.e., ρ slightly larger than 1):

$$f(\rho) \approx \left[\frac{9}{4} \right]^{\frac{2}{3}} (\rho - 1)^{\frac{4}{3}} \qquad (3.35)$$

We can use this approximation to start our numerical differential equation solution at a non-singular point (like $\rho = 1.00001$).

Real devices are designed with $b/a \gg 1$. The behavior of f for large ρ can be determined by finding A and α for which $f = A\rho^\alpha$ is a solution to the differential equation. One finds:

$$f = \left(\frac{9}{4}\,\rho \right)^{\frac{2}{3}} \qquad (3.36)$$

A useful approximation for the range: $100 < b/a < 1000$ is:

$$f = \left(\frac{9}{4}\,\rho \right)^{\frac{2}{3}} + 2 \qquad (3.37)$$

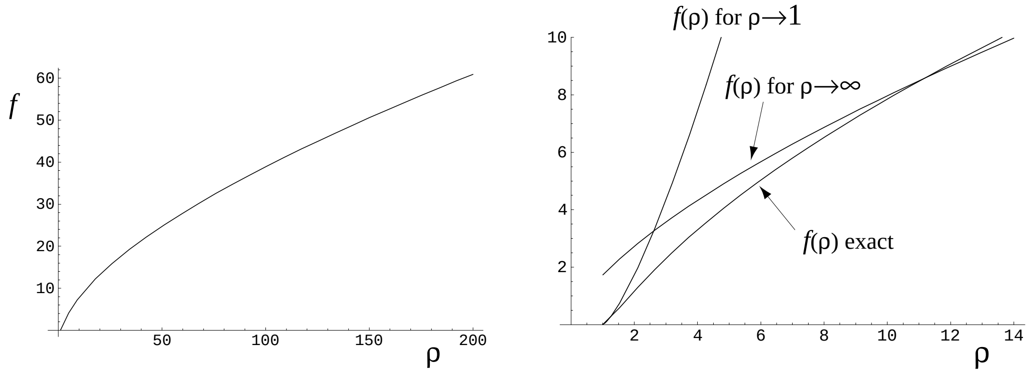

Figure 3.3: The plot on the left displays the dimensionless voltage f obtained by numerical solution to the differential equation. The plot on the right compares various approximations for f to this numerical solution.

(For example, the device used in lab has $b/a = 121.5$. For this value, the differential equation gives $f = 44.136$; the above approximation gives: $f = 44.130$.)

We recover Child's law by rearranging (3.30):

$$\frac{2\pi\epsilon_0\sqrt{\frac{2e}{m}}}{a}\left[\frac{V_A}{f(b/a)}\right]^{\frac{3}{2}} = j = I/\ell \tag{3.38}$$

Note: Langmuir's original work (Phys. Rev. **22**, 347 (1923)) on this subject is expressed in terms of β where:

$$\beta^2(\rho) \equiv \frac{4}{9}\,\frac{f^{\frac{3}{2}}}{\rho} = \begin{cases} \underset{\rho\to 1}{\longrightarrow} & (\rho-1)^2 \\ \underset{\rho\to\infty}{\longrightarrow} & 1 \end{cases} \tag{3.39}$$

So:

$$\boxed{\frac{8\pi\epsilon_0\ell\sqrt{\frac{2e}{m}}}{9b\beta^2}\,V_A^{\frac{3}{2}} = I} \tag{3.40}$$

$\beta^2 = 1.072$ for the device used in lab.

Richardson's Law

Most any thermal process is governed by the Boltzmann factor:

$$\exp\left(-\frac{\Delta E}{kT}\right) = e^{-\Delta E/kT} \tag{3.41}$$

where k is the Boltzmann constant. Approximately speaking the Boltzmann factor expresses the relative probability for an event requiring energy ΔE in a system at (absolute) temperature T. Clearly if $\Delta E \gg kT$, the probability of the event happening is low. If an electron requires an energy W (called the work function) to escape from the metal, The

Boltzmann factor suggests that this would happen with relative probability $e^{-W/kT}$. Thus you should expect that the current emitted by a heated metal would follow:

$$I \sim e^{-W/kT} \tag{3.42}$$

Clearly you should expect different elements to have different work functions, just as different atoms have different ionization potentials. What is surprising is that the proportionality factor in the above equation includes a universal constant — that is, a constant that just depends on the properties of electrons (and, most importantly, Planck's constant, h) and does *not* depend on the type of material. (This situation is similar to that of blackbody radiation, in which photons rather than electrons are leaving a heated body, which was Planck's topic in discovering his constant. We will take up this topic on page 53.) Thermionic emission probes the quantum state of the electrons statistically, whereas the photoelectric effect probes much the same physics electron by electron. (The next level problem is to explain why this universal constant (the Richardson constant, A) in fact does depend a bit on the material.) To show:

$$J = AT^2 e^{-W/kT} \tag{3.43}$$

where

$$A = \frac{4\pi emk^2}{h^3} = 1.2 \times 10^6 \mathrm{A/m^2K^2} \tag{3.44}$$

Quantum Theory: Free Electron Gas

Instead of thinking about electron particles bouncing around inside a box, de Broglie invites us to consider standing waves of electron probability amplitude:

$$\psi = N \exp(ik_x x) \exp(ik_y y) \exp(ik_z z) = N e^{i\mathbf{k}\cdot\mathbf{r}} \tag{3.45}$$

Recall [5] that vector $\hbar\mathbf{k}$ is the momentum, $\mathbf{p} = m\mathbf{v}$, of the electron and $\hbar = h/2\pi$. Periodic boundary conditions on the box (which we take to be a cube with one corner at the origin and the diagonally opposite corner at the point $\mathbf{r} = (L, L, L)$) require each component k_i to satisfy:

$$k_i = \frac{2\pi\, n_i}{L} \tag{3.46}$$

where each n_i is an integer. Thus each wave function is specified by a triplet of integers: $\mathbf{n} = (n_x, n_y, n_z)$, the $\mathbf{n}$-vector. Applying Schrödinger's equation, we find that this wavefunction has energy:

$$E(\mathbf{n}) = \frac{\hbar^2\mathbf{k}^2}{2m} = \frac{(2\pi\hbar)^2\mathbf{n}^2}{2mL^2} = \frac{(2\pi\hbar)^2(n_x^2 + n_y^2 + n_z^2)}{2mL^2} \tag{3.47}$$

Notice that there is a definite relationship between the velocity vector of the electron and the $\mathbf{n}$-vector.

$$\mathbf{v} = \frac{2\pi\hbar}{mL}\,\mathbf{n} \tag{3.48}$$

Another way of saying the same thing is that allowed quantum-state velocities form a cubic lattice with cube-side $2\pi\hbar/mL$. The number of states with electron velocities in some specified region (for example a velocity-space parallelepiped with sides: $\Delta v_x \Delta v_y \Delta v_z$) can

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[5] For a review see: `http://britneyspears.ac/physics/dos/dos.htm`

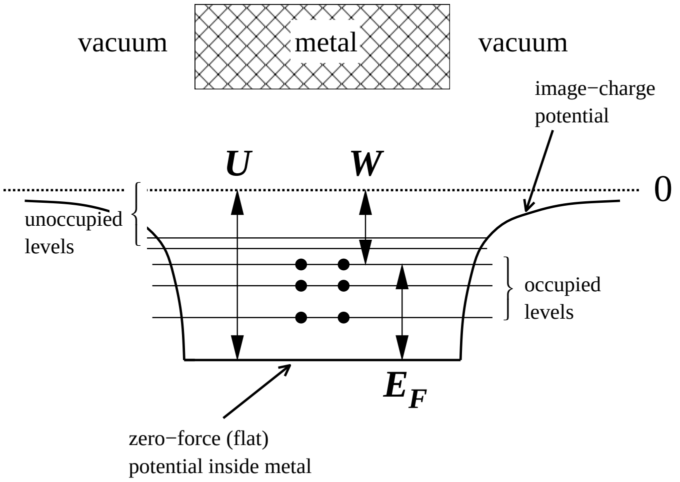

Figure 3.4: Electrons in the metal experience a constant confining potential of depth U. Possible quantum mechanical states for these electrons are displayed as horizontal lines. Electrons fill all the available states up to the Fermi energy, E_F. The work function, W, is defined at the minimum energy needed to remove an electron from the metal. As shown above: $W = U - E_F$.

be found from the number of $2\pi\hbar/mL$ sided cubes that fit into the region, which is the volume of that velocity-space region divided by $(2\pi\hbar/mL)^3$. Hence:

$$\text{number of states with velocity between } \mathbf{v} \text{ and } \mathbf{v}+\Delta\mathbf{v} = \frac{\Delta v_x \Delta v_y \Delta v_z}{(2\pi\hbar/mL)^3} \tag{3.49}$$

$$\text{number of states } \underline{\text{per volume}} \text{ with velocity between } \mathbf{v} \text{ and } \mathbf{v}+\Delta\mathbf{v} = \frac{\Delta v_x \Delta v_y \Delta v_z}{(2\pi\hbar/m)^3}$$

$$= \left(\frac{m}{2\pi\hbar}\right)^3 \Delta v_x \Delta v_y \Delta v_z = \mathcal{N} \Delta v_x \Delta v_y \Delta v_z \tag{3.50}$$

where $\mathcal{N}$ is the (constant) density of states in velocity space.

Quantum Theory: Fermi Energy

Fermions (half-integer spin particles), in contrast to bosons (integer spin particles), cannot group together. Since the electron is “spin $\frac{1}{2}$”, each of the above states can hold at most 2 electrons: one spin up and one spin down. The probability that a particular fermion state with energy E will be occupied is given by a generalization of the Boltzmann factor called Fermi-Dirac statistics:

$$f(E) = \frac{1}{1+\exp\left(\frac{E-E_F}{kT}\right)} \tag{3.51}$$

where E_F is called the Fermi energy. The Fermi energy is basically a disguise for the number of electrons, as, approximately speaking, it is the dividing line between occupied states and unoccupied states. (If the Fermi energy is high, there must be lots of occupied states and hence lots of electrons.) Note that if $E \gg E_F$, the exponential factor is huge and we can

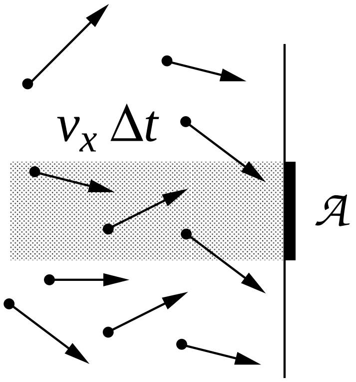

Figure 3.5: Consider just those electrons with some particular x-velocity, v_x. In order to hit the wall during the coming interval Δt, an electron must be sufficiently close to the wall: within $v_x\Delta t$. The number of such electrons that will hit an area $\mathcal{A}$ will be equal to the number of such electrons in the shaded volume (which is the perpendicular extension of $\mathcal{A}$ a distance $v_x\Delta t$ into the volume). Note that many electrons in that volume will *not* hit $\mathcal{A}$ because of large perpendicular velocities, but there will be matching electrons in neighboring volumes which will hit $\mathcal{A}$. To find the total number of hits, integrate over all possible v_x.

neglect the "+1" in the denominator so

$$f(E) \approx \exp\left(-\frac{E-E_F}{kT}\right) \tag{3.52}$$

that is, if $E \gg E_F$ Fermi-Dirac statistics approximate the Boltzmann factor.

Classical Theory: Electron Escape

The density of states combined with the Boltzmann factor gives us the number of free electrons per unit volume with a particular velocity. In order for an electron to escape during some time Δt, it must have v_x sufficient to overcome the image-charge barrier and it must be sufficiently close to the wall. All the electrons with $v_x > \sqrt{2U/m}$ within a distance $v_x\Delta t$, will escape, where U is the depth of the potential well for the electrons. Thus the number of electrons escaping through area $\mathcal{A}$ during Δt is:

$$\begin{aligned}
&\int_{\sqrt{2U/m}}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \; 2\mathcal{N} f(E)\; \mathcal{A} v_x \Delta t \\
&\quad = 2\mathcal{N} e^{E_F/kT} \mathcal{A}\Delta t \int_{\sqrt{2U/m}}^{\infty} e^{-mv_x^2/2kT}\; v_x dv_x \int_{-\infty}^{\infty} e^{-mv_y^2/2kT} dv_y \int_{-\infty}^{\infty} e^{-mv_z^2/2kT} dv_z \\
&\quad = \frac{4\pi m (kT)^2}{(2\pi\hbar)^3}\, \mathcal{A}\Delta t\, \exp\left(-\frac{(U-E_F)}{kT}\right) \qquad (3.53)
\end{aligned}$$

where we have used the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-\alpha z^2} dz = \sqrt{\frac{\pi}{\alpha}} \tag{3.54}$$

Table 3.1: G.E. FP-400 Specifications

Filament (W) length	3.17 cm	(1.25")
Filament diameter	0.013 cm	(0.005")
Anode (Zr coated Ni) I.D.	1.58 cm	(0.620")
Maximum filament voltage	4.75 V	
Maximum filament current	2.5 A	
Maximum anode voltage	125 V	
Maximum anode current	55 mA	
Maximum anode dissipation	15 W	

The electric current density is the electric *charge* escaping per time per area:

$$J = \frac{e \times \text{number escaping}}{\mathcal{A}\Delta t} = \frac{4\pi e m (kT)^2}{h^3} \exp\left(-\frac{W}{kT}\right) \tag{3.55}$$

which is Richardson's equation, with work function W given by $U - E_F$.

Experiment: Richardson's "Constant" is Material Dependent!

Experimentally it is found that Richardson's constant depends on the material[6] Why?

1. The work function depends on temperature (due to, for example, thermal expansion of the lattice of atoms). If the data analysis assumes it's constant, the resulting A will be grossly in error.

2. Classically reflection requires a turning point (where $v_x = 0$), whereas quantum mechanical reflections are possible just due to sharp changes in potential. Quantum mechanical reflection at the metal boundary was not included in our calculations; we assumed every energetic electron headed toward the wall went through the wall.

3. Surface contamination can affect emission probability. In fact, it was originally thought that thermionic emission was 100% due to surface contamination. (You can think of surface contamination as a locally varying work function.)

4. Even in the absence of surface contamination, in typical experiments, the metal is polycrystalline and different crystal surfaces have different work functions.

Experiment

This experiment involves thermionic emission from the hot tungsten filament of a G.E. FP-400 vacuum tube.

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[6] This should remind you a bit of the material dependent emissivity, ϵ_T, for blackbody radiation to be discussed on page 53.

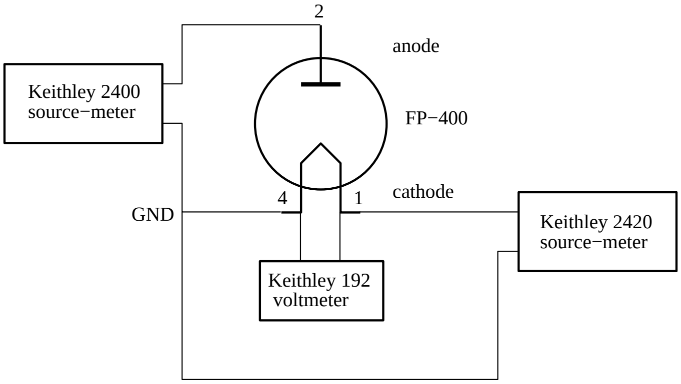

Figure 3.6: Circuit diagram (note tube pin labels) for thermionic emission experiment.

Temperature Determination

Often temperature measurement in physics experiments is easy. In the "normal" range of temperatures there are many types of transducers which convert temperature to an electrical quantity (e.g., Pt resistance thermometers, thermocouples, thermistors, ICs). However at the extremes of high and low temperature, measurement becomes tricky. Questions like "What exactly defines *temperature*?" must then be answered. This experiment requires "high" temperatures in a vacuum, so we do not have direct contact with the material whose temperature we seek. In addition the FP-400 tube was not built by us, so we have limited direct information about the device.

One common way to measure temperature is by using the temperature dependence of resistance. The resistance of metals is approximately proportional to temperature. Jones and Langmuir[7] have published a table of resistance vs. temperature for tungsten, from which Kirkman has found an approximating formula:

$$T_r = 112 + 202x - 1.81x^2 \tag{3.56}$$

where x is the ratio of the hot resistance to that at 293 K.

A problem with this approach is that the measured resistance, R_{measured}, will include both the resistance of the tungsten[8] filament, R_W and the wires supporting it in the vacuum tube, R_{support}. Thus the quantity we seek (tungsten filament resistance, R_W) must be calculated as the small difference between two numbers:

$$R_W = R_{\text{measured}} - R_{\text{support}} \tag{3.57}$$

a situation that produces big relative errors. Additionally, we have no independent way of measuring R_{support} (we can't take the tube apart); In the end you will measure R_{support} at room temperature and then assume it is constant[9].

There is a further problem with any measurement of *voltage* when parts of the system are at different temperatures: thermally induced emfs (thermocouples). If the ends of the tungsten filament are at different temperatures, there will be a voltage induced approximately

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[7] GE Rev **30**, 310 (1927)

[8] The chemical symbol for tungsten is W from the German Wolfram

[9] See "Assuming Away Variation" page 18 in Chapter 0

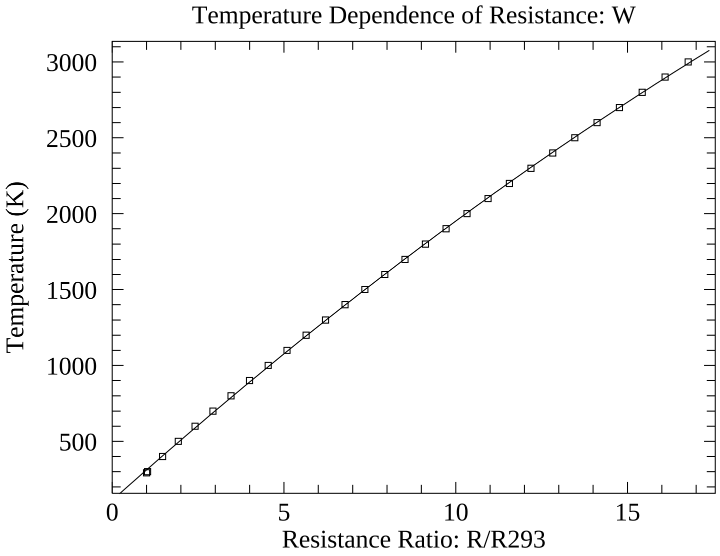

Figure 3.7: The temperature dependence of tungsten-filament resistance from the data of Jones & Langmuir with an approximating curve. The x-axis is the ratio of the hot resistance to that at 293 K.

proportional to the temperature difference between the two ends. This additional voltage source confuses the resistance determination. The phenomena can be detected and corrected by reversing the direction of current flow (which reverses the Ohm's law voltage, but does not affect the sign of the thermal voltage.) Thermal voltages are generally less than a mV, and so are negligible once our measured voltages approach one volt.

Another approach is suggested by Jones & Langmuir. In a vacuum the temperature of a current-carrying wire is the result of an equilibrium between electrical power dissipated in the wire and energy lost in the form of radiation. (We assume that energy lost through conduction — either through the wire-supports or residual air in the "vacuum" — is negligible.) According to the Stefan-Boltzmann law, the power radiated from the surface of a hot black-body is given by:

$$P = \sigma T^4 \mathcal{A} \tag{3.58}$$

where σ is the Stefan-Boltzmann constant, T is the temperature of the body, and $\mathcal{A}$ is the surface area of the body. (In fact tungsten is not a black-body, so when applied to tungsten the above should be multiplied by a "fudge factor", the total emissivity ϵ_T, about 0.3.) Using just crude approximations, we equate the electrical power dissipated to the power radiated:

$$I^2 T \frac{\ell}{d^2} \sim I^2 T \frac{\ell}{\frac{\pi}{4} d^2} \sim I^2 R = \epsilon_T \sigma T^4 \pi d \ell \sim T^4 d\ell \tag{3.59}$$

where d is the diameter of the wire and ℓ is the length of the wire. On the right hand side we've assumed that the resistivity of tungsten is proportional to temperature, and on the left hand side we've assumed ϵ_T doesn't depend on temperature. We conclude:

$$I^2 \sim T^3 d^3 \tag{3.60}$$

$$\left[\frac{I}{d^{\frac{3}{2}}}\right]^{\frac{2}{3}} \sim T \tag{3.61}$$

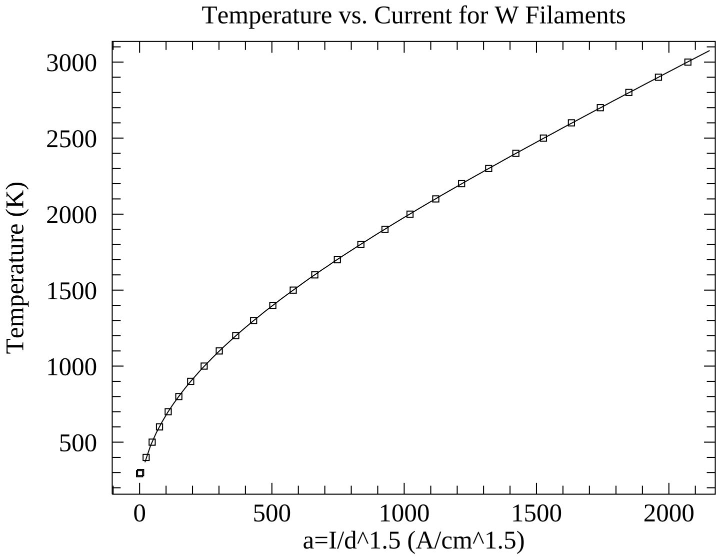

Figure 3.8: The temperature of an in-vacuum tungsten filament as a function of current from the data of Jones & Langmuir with an approximating curve. The x-axis is the current divided by the diameter of the wire (in cm) raised to the $\frac{3}{2}$ power.

Absent the above approximations we can hope that temperature is a function of $a \equiv I/d^{\frac{3}{2}}$.

Once again Jones & Langmuir provide us with calibrating data for this expected relationship. For temperatures 400 K $< T <$ 3000 K, Kirkman finds:

$$T_i = 117 + 56a^{0.5} + 0.00036a^{1.8} \tag{3.62}$$

Finally, attaining thermal equilibrium[10] is a problem that affects most any temperature measurement. The balance between electrical power in and heat lost is not immediately achieved. The parts of the system with large heat capacities (particularly the filament supports and other large structures in the tube), will only gradually approach equilibrium. Thus "soak" time is required following each jump in heating power. The effect of this "thermal inertia" is seen in "hysteresis": temperatures obtained while increasing the temperature disagree with those found while decreasing the temperature. This will be an important source of uncertainty.

Hands-on Electrical Measurements

Support Resistance: R_{support}

As shown in Figure 3.6, the tungsten filament (cathode) is powered by a Keithley 2420 source-meter. The filament+support voltage is measured directly at the socket with a Keithley 192 voltmeter. By combining the current through the filament (measured from the 2420) with the voltage across the socket (from the 192), a resistance, R_{measured}, can

---

[10] See "Special Problem: Temperature" page 17 in Chapter 0

be determined. At room temperature, the filament resistance, R_W, can be calculated from the filament geometry (see Table 3.1) and the resistivity of W-filament material at room temperature: $\rho_{293} = 5.49\ \mu\Omega\cdot\text{cm}$. (You should calculate: $R_W \sim .1\ \Omega$.) Then:

$$R_{\text{support}} = R_{\text{measured}} - R_W \tag{3.63}$$

Because room temperature R_{measured} is 'small' (and hence error prone), you will make three distinct measurements of it. Begin by sourcing 1 mA and then 10 mA into the room temperature filament (using the 2420), reading the resulting voltages on the 192, and calculating the corresponding R_{measured}. Follow up those two measurements with a four-terminal resistance measurement just using the 192. (If R_{measured} is substantially larger than $\sim .2\ \Omega$, confirm that you have firm, low-resistance contacts between the socket and the tube. Working the socket-tube contact may be required to find the lowest resistance spot.)

Maximum Filament Current, 2420 Voltage Compliance Limit

Following your determination of R_{support} in a room temperature tube, check out the conditions required to stay just below the tube's maximum ratings (4.75 V, 2.5 A). Using the 2420, successively source filament currents of 2.0 A, 2.1 A, 2.2 A, ... to directly determine the maximum current you can use without exceeding the 4.75 V limit across the tube's filament. Note that at just below maximum conditions, the 2420 will probably need to produce voltages above 4.75 V (because of the resistance of the external wires: the voltage drop across the connecting wires is not zero). Record the maximum 2420 voltage and tube current allowed by the tube's ratings; you will need these numbers in step #2 of your computer program.

Data Collection Plan

You will be collecting two types of data at the same time: thermal characteristics of the filament and the thermionic properties of the tube (anode current/voltage relationship). Starting at a filament current of 0.9 A, increase the current flowing through the filament in steps of 0.1 A to a maximum current (found as described above, about 2.4 A) and then reverse those steps down to a filament current 1.0 A. The up-sweep in filament current followed by the down-sweep will allow you to test for hysteresis. At each step in current, allow the filament to approach thermal equilibrium (wait, say, 15 seconds) and then measure the voltage across and current through the cathode/anode. Calculate filament temperature two ways (Equations (3.56) and (3.62)). Average the two to estimate *the* temperature, and use half the absolute value of the difference to estimate the uncertainty.

You see above a classic example of systematic error. The temperature is measured two different ways. Direct application of error propagation formulas to these temperatures calculated from 6-digit meter values would suggest small uncertainties. However the two temperatures in fact disagree. If only one method had been used to measure temperature, we would have badly underestimated the error.

T^4 vs. Power: Testing Stefan-Boltzmann

By conservation of energy we know that the power dumped into the filament (mostly from electrical heating, but also from other sources like radiation from the room temperature environment to the filament) should equal the power out of the filament (from black-body radiation and other factors like conduction down the supports). Thus:

$$\begin{aligned} \epsilon_T \sigma \mathcal{A} T^4 &= I^2 R_W + \text{ constant} && (3.64) \\ T^4 &= \frac{1}{\epsilon_T \sigma \mathcal{A}}\, I^2 R_W + \text{ constant} && (3.65) \\ y &= bx + a && (3.66) \end{aligned}$$

A graph of T^4 vs. power should be a straight line from which you will determine ϵ_T. (Note that the error in power is quite small, so we have properly used it as an x variable.) In order to test this relationship you will want to make a file containing the filament power, T^4 (use the average of the two temperatures: $(T_i^4 + T_r^4)/2)$, and the error in T^4 (use half the difference from the two temperatures: $|T_i^4 - T_r^4|/2)$.

I_A vs. V_A: Testing Child and Richardson

You will collect anode current vs. voltage curves for each different filament (cathode) temperature. Use the Keithley 2400 to sweep the anode voltage logarithmically from 2 V to 120 V. (Note the maximum limits for the anode: 0.055 A or 125 V. Do not exceed either!) According to Child's law, the anode current, I_A, should increase in proportion to $V_A^{\frac{3}{2}}$. Of course, at sufficiently high voltage the current will be limited by the maximum electron evaporation rate, and a current plateau forms at a level given by Richardson's law. At the maximum filament current (corresponding to the maximum filament temperature and evaporation rate), plateau formation occurs at very high voltage and you have the longest run of data following Child's law. Make a file containing V_A, I_A, and δI_A which you can use to fit to the Child's law functional form:

$$I_A = k_1 \, (V_A - k_2)^{\frac{3}{2}} \tag{3.67}$$

In addition, you will want to make a big continuous file containing: V_A, I_A at every temperature tested. The current plateaus can be extracted from this file and fit to the Richardson relationship:

$$I_A = k_1 \, \mathcal{A} T^2 \, e^{-k_2/T} \tag{3.68}$$

Computer Data Collection

As part of this experiment you will write a computer program to control the experiment. *Plagiarism Warning*: like all lab work, this program is to be your own work! Programs strikingly similar to previous programs will alarm the grader. I understand that this will often be a difficult and new experience. Please consult with me as you write the program, and test the program (with tube disconnected!) before attempting a final data-collecting run.

Your program will control all aspects of data collection. In particular it will:

0. Declare and define all variables.

1. Open (i.e., create integer nicknames—i.e., `iunit`—for) the enets `gpib0` and `gpb1`.

2. Initialize meters—requires knowing the gpib primary address—i.e., `iadd`—of the meter and the `iunit` it is attached to. Get the status of each meter after you have initialized it.

   (a) Each source-meter must be told the maximum voltage and current it must produce during the experiment. Initialize the 2400 (anode voltage/current) for the tube maximum ratings.

   (b) Initialize the 2420 (filament voltage/current) for the near tube-maximum ratings found above[11]. In the following I assume the current maximum is 2.4 A, but it may be different for your tube.

   (c) Initialize the 192 for autorange DC voltage measurements.

3. Open the files:

   (a) `filament.dat` (intended for: I_f, V_f, T_r, T_i of filament).

   (b) `stefanB.dat` (intended for: power, T^4, δT^4 of filament).

   (c) `VI.dat` (intended for: all V_A, I_A of anode, with comments ('!') for filament I_f, T_r, T_i).

   (d) `child.dat` (intended for: V_A, I_A, δI_A of anode at maximum filament current).

   (e) `child-.dat` (like above but intended for a downsweep of anode voltage).

   (f) `rich.dat` (intended for: T_i, T_r, I_A — i.e., the estimated temperatures and the corresponding maximum anode current for Richardson's Law).

4. Tell the 2420 source-meter to source a filament current of 0.9 A.

5. Let the system sleep for 60 seconds to approach thermal equilibrium.

6. Do a sequence of measurements where the filament temperature is sequentially *increased* (i.e., a temperature up-sweep) with filament currents ranging from 0.9 A to some maximum (e.g., 2.4 A) current in steps of 0.1 A.

   (a) Tell the 2420 source-meter to source a filament current (I_f).

   (b) Let the system sleep for 15 seconds to approach thermal equilibrium.

   (c) Request a logarithmic sweep of the anode voltage (controlled by the 2400 source-meter) from 2 V to 120 V. Receive the resulting arrays: V_A and I_A.

   (d) Turn off the anode voltage.

   (e) Repeat (a) thus receiving an updated version of the filament current (it will be very close to the requested current).

   (f) Read the 192 to get the filament voltage (V_f).

   (g) Using Eqs. (3.56) and (3.57), calculate T_r based on the calculated tube resistance R_{measured}, the calculated room temperature filament resistance and R_{support}.

   (h) Calculate T_i from Eq. (3.62)

---

[11] See Hands-on Electrical Measurements, p. 54. Recall: the 2420 maximum voltage will need to be a bit above 4.75 V. If you have not yet completed those measurements, temporarily initialize with 4.75 V.

   (i) Write a line to the file `filament.dat` reporting: I_f, V_f, T_r, T_i.
   (j) Write a line to the file `stefanB.dat` reporting: filament power ($R_W I_f^2$), T^4, and δT^4 (see p. 56).
   (k) Write a comment line (i.e., starts with '!') to the file `VI.dat` reporting filament data: I_f, T_r, T_i.
   (l) Write the anode sweep data (one data-pair per line) in the file `VI.dat`.
   (m) Write a line to the file `rich.dat` reporting T_i, T_r, I_A. Use I_A at V_A=120 V as the estimated plateau current. (When the experiment is over, you will need estimate δI_A based on hysteresis, and may need to delete I_A values if, for example, the current did not plateau or if cold emission substantially added to the plateau current.)
   (n) Increment the filament current by 0.1 A and return to (a).

7. Collect data for Child's Law. Begin by repeating a normal anode sweep at the maximum filament current; follow all the steps (a)–(m) outlined in 6 above. In addition, write the anode sweep data (V_A, I_A, δI_A) in the file `child.dat` (one data-triplet per line). In this case, δI_A will be calculated from the manufacturer's specs: percent+digits; the fortran function `eAk2400` can do this error calculation automatically. Now check for hysteresis by doing a reverse anode sweep: from 120 V down to 2 V. Write this reverse anode sweep data (V_A, I_A, δI_A) in the file `child-.dat`.

8. Do a sequence of measurements where the filament temperature is *decreased* (i.e., a temperature down-sweep) by sequentially sourcing filament currents from one step down from maximum (e.g., 2.3 A) back to 1.0 A. Follow steps (a)–(m) used in the temperature up-sweep (part 6 above) and then:

   (n) decrement the filament current by 0.1 A and return to (a).

9. Turn off the output of the 2420.

10. Close all files.

Note that the 0.9 A filament current data is just trash collected so the 1.0 A filament current data is taken on a pre-warmed filament.

Observations

While the computer collects the data observe the light from the filament. (There is a $\frac{1}{16}$" diameter hole in the anode allowing light from the mid-point of the filament to escape.) Initially the filament will not appear to be incandescent (i.e., not a source of light at all: dark) so it may help to turn off the lab lights to better observe the beginning of visible incandescence. According to the Stefan-Boltzmann law the light energy radiated depends on T^4, so small changes in T produce big changes in light intensity. In addition to changes in light intensity, you should look for the more subtle change in light color: from a dull red to a brilliant yellow. Record your observations! At what filament temperature did you first see the filament producing light?

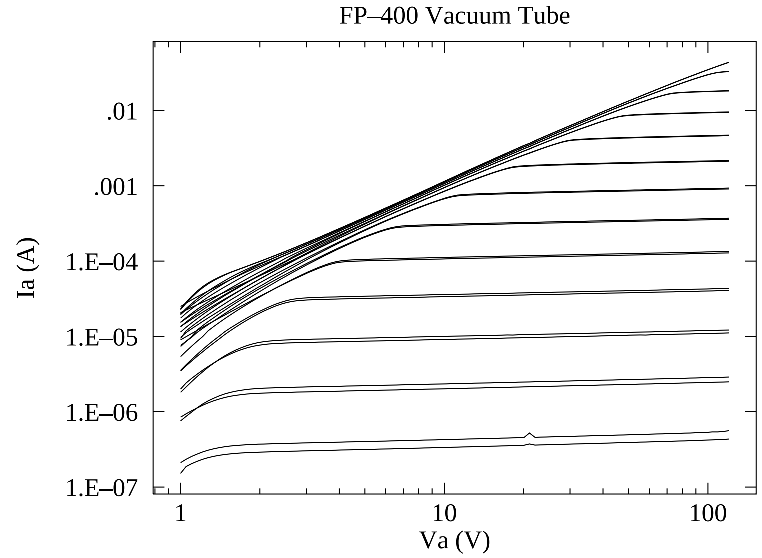

Figure 3.9: The temperature dependence of thermionic emission in a FP-400 vacuum tube. Each curve shows the anode current-voltage relationship at a particular filament temperature. At high filament temperatures and low anode voltages the anode current follows Child's law: an upward sloping straight line on this log-log plot. At sufficiently high anode voltage, the filament current plateaus at a value given by Richardson's law. At low filament temperatures and high anode voltages, "cold emission" may augment (distort) the plateau.

Data Analysis

Beginnings

The main result of this experiment is a plot similar to Figure 3.9 showing the anode current-voltage relationship at various filament temperatures. Production of such a multi-plot is a bit complex, and you will almost certainly need further detailed instructions from your instructor on using the program `Nplot`. Note that since this is a log-log plot, negative anode currents must also be edited out of `VI.dat` to make this plot.

It may be difficult to determine the plateau level for the lowest and highest filament temperatures, in which case you must edit out those data points in `rich.dat`. **Note:** If the high-temperature VI sweep reaches a plateau, then Child's Law will not apply at high V_A so `child.dat` will require editing. If it does not reach a plateau, then Richardson's Law does not apply so `rich.dat` will require editing. In Figure 3.9, I see no sign of a plateau at the maximum filament current, but in the range $I_f = 1.1$ to 2.3 A, plateau currents can be determined. (I.e., I removed from `rich.dat` the non-plateau data point for $I_f = 2.4$ A, and also the aberrant initial data for $I_f = 1.0$ A.)

Additionally a glance at the file `rich.dat` should demonstrate systematic temperature error: $T_i \neq T_r$. Clearly fits assuming $T = T_r$ will produce different results from fits assuming $T = T_i$. Absent further information, our best estimate for T must be something like the

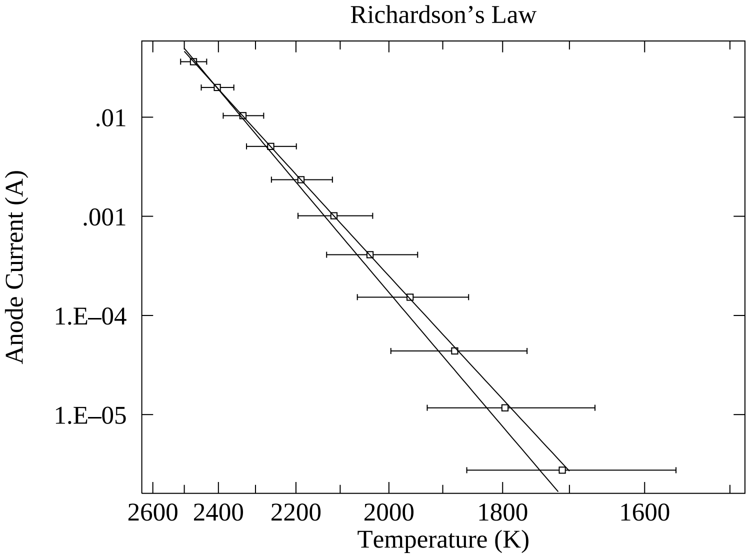

Figure 3.10: A Richardson Plot of thermionic emission in a FP-400 vacuum tube. Each data point displays the plateau anode current at a particular filament temperature. The curve through the data fits for the work function; the slightly steeper curve uses the book value for the work function.

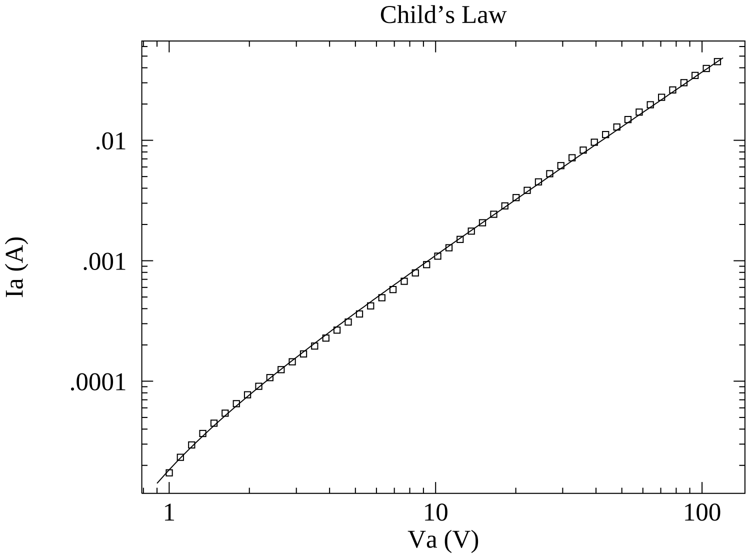

Figure 3.11: A plot of the space-charge limited thermionic emission in a FP-400 vacuum tube (Child's law). The data was taken at a filament current of 2.4 A. Every other data point has been eliminated so the fit line is not obscured. Note that the fit line systematically misses the data, sometimes a bit high others a bit low. The measurement errors are tiny, so these small misses do result in a "too-large" χ^2. Nevertheless, the law provides an excellent summary of the data over a huge range of variation.

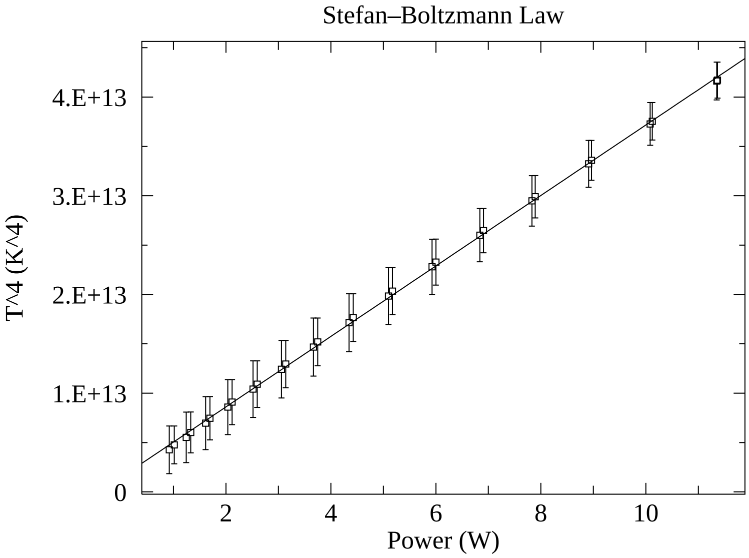

Figure 3.12: A test of the Stefan-Boltzmann law: that power radiated is proportional to T^4. Note that the fit line hits well within each error bar. The χ^2 for this fit will be "small". Evidently the average temperature is a better measure of temperature than you might expect from the deviations between T_i and T_r.

average of T_i and T_r. The difference between fit parameters produced assuming $T = T_r$ and those produced assuming $T = T_i$ will allow use to estimate the systematic error in those parameters.

Now according to Richardson's law, these plateau currents should satisfy:

$$I = \mathcal{A}AT^2\, e^{-W/kT} = k_1\, \mathcal{A}T^2\, e^{-k_2/T} \tag{3.69}$$

where $\mathcal{A}$ is the tungsten filament surface area. However, since we are currently only seeking error estimates, simplified analysis is justified. Clearly the largest error is in T, so the standard approach would be to put T on the y-axis and I on the x-axis: the opposite of what is implied by the above equation. However, we can't simply solve the above equation for T without some seemingly huge approximations. It turns out that $e^{-k_2/T}$ is the significant factor in the above equation, so we start by ignoring the T^2 and assume:

$$I = a\, e^{-b/T} \tag{3.70}$$

Now if we take ln of both sides:

$$\ln(I) = \ln(a) - k_2\, \frac{1}{T} \tag{3.71}$$

or

$$\frac{1}{T} = \ln(a)/k_2 - \frac{1}{k_2}\, \ln(I) \tag{3.72}$$

This equation is now in a form[12] known to *WAPP*$^+$. Thus you can quickly *WAPP*$^+$ (I, T_i) and (I, T_r) data from `rich.dat` (for simplicity assume no x-error and no y-error), and

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[12] Inverse-Natural Log: $1/y = A + B\log(x)$

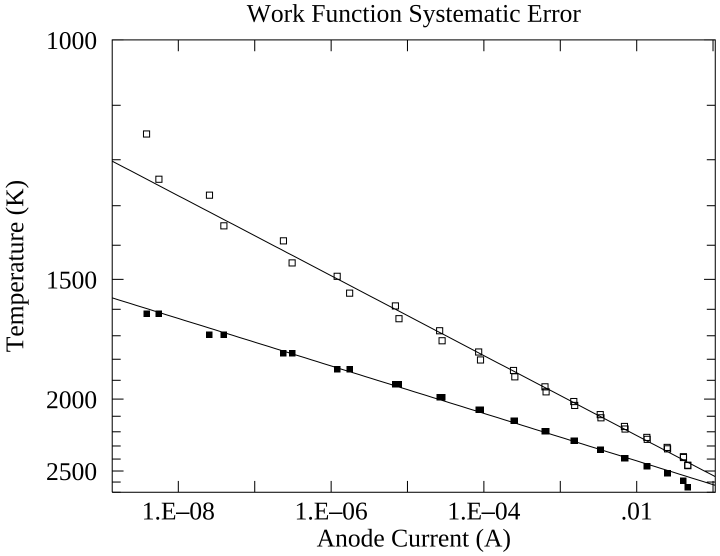

Figure 3.13: Simplified analysis suggests Richardson's Law data can be fit to the Inverse-Natural Log relationship of Eq. 3.72. The filled-square points use $y = T_i$, the unfilled-square points use $y = T_r$. Systematically different temperature measurements yield systematically different B (slopes in the above plot) and hence systematically different work functions $W = -k/B$. Note the inexact pairing of the data due to temperature hysteresis.

determine the range of $k_2 = -1/B$ consistent with the differing temperature data. Request *WAPP*$^+$ make a linearized plot of your data with x-scale: log and y-scale: inverse as in Fig. 3.13. This should allow you to check for aberrant data points. (Usually the high temperature curve has not plateaued, and so the high temperature data point must be removed. Occasionally the low temperature curve is also aberrant.)

The range for $W = k_2 k$ has been found by simplified fits; the best value for W will be found (see following) with a fit to the proper functional form with due account for the uncertainty in current caused by hysteresis and the best estimate for the actual temperature.

The Stefan-Boltzmann Law is the easiest law to check: you can quickly *WAPP*$^+$ the data in the file `stefanB.dat` to produce a plot similar to Fig. 3.12. I expect you'll find a small reduced χ^2 due to the large systematic error in temperature, so a bit of additional work will be required to estimate $\delta\epsilon_T$.

Child's Law

At sufficiently high filament temperature (and low anode voltage), Child's law governs the anode current-voltage relationship:

$$I_A = \frac{8\pi\epsilon_0 \ell \sqrt{\frac{2e}{m}}}{9b\beta^2} V_A^{\frac{3}{2}} \tag{3.73}$$

(see vicinity of Eq. 3.40, page 47, for a definition of symbols) At the highest filament temperature (e.g., filament current of 2.4 A) you have saved the $(V_A, I_A, \delta I_A)$ data in the file `child.dat`. Now fit and plot this data to the functional form:

$$I_A = k_1 \left(V_A - k_2\right)^{\frac{3}{2}} \tag{3.74}$$

(where k_2 represents an offset between ground and the actual average voltage of the filament), producing a result similar to Fig. 3.11. Do not be surprised if you get a huge reduced χ^2. Find an estimate for k_1 error either by a 'fudged fit' or a bootstrap.

Follow exactly the same process for the downsweep data in the file `child-.dat`. Generally you will find that the k_1 values for the two sweeps differ by more than computer-based value of δk_1. Systematic error (here a type of hysteresis) is responsible. Note that the usual reduced χ^2 alerts us to a problem, but measuring twice (in different ways) provides an estimate (perhaps still an under-estimate) for δk_1: half the difference between the two values of k_1.

We expect that

$$k_1 = \frac{8\pi\epsilon_0 \ell \sqrt{\frac{2e}{m}}}{9b\beta^2} \tag{3.75}$$

so with known geometry, the electron charge-mass ratio e/m can be calculated. Since the FP-400 is a finite length cylinder (rather than the infinite cylinder discussed in our theory) use the effective length[13] $= 0.7 \times \ell$ as the length of the cylinder. But what can we use as errors for the 'book' values for b and ℓ? Thermal expansion of the filament should, by itself, change ℓ by about 1% (notice the spring-tensioned support for the filament in the FP-400). Based on the sigfigs in the reported values, I estimate:

$$\frac{\delta(b/\ell)}{(b/\ell)} \approx 3\% \tag{3.76}$$

Calculate e/m and its error. Compare to the 'known' value (citation required!).

Stefan-Boltzmann Law

You should have already checked for an approximately linear relationship between electrical power in and T^4:

$$\text{Power} = \epsilon_T \sigma \mathcal{A} T^4 \tag{3.77}$$

and found a reduced χ^2 indicative of large systematic uncertainty in temperature. We now seek an estimate for ϵ_T (with error) at the highest temperature. The best value for ϵ_T can be found by plugging in our best estimates for T^4 and the measured power (found in the file `stefanB.dat`), and $\mathcal{A}$ (calculated based on the dimensions recorded in Table 3.1). Alternatively ϵ_T could be calculated based on slope as suggested by Eq. 3.66. But how should we incorporate the large systematic errors in T^4 and the unknown systematic error in $\mathcal{A}$? For the surface area $\mathcal{A}$, all we know is the 'book' values for the dimensions of the filament. Based on the sigfigs in the reported values, I estimate:

$$\frac{\delta\mathcal{A}}{\mathcal{A}} \approx 10\% \tag{3.78}$$

---

[13] This effective length corrects for current loss through the ends of the cylinder under space-charge situations. A smaller correction should be applied when large anode voltages are designed to sweep up all evaporated electrons, i.e., for Richardson's Law, where 90% electron collection seems appropriate. The details of handling such corrections to theory may be found in reference 1.

(mostly due to the filament diameter, where a 'small' uncertainty leads to a large fractional uncertainty). We can then use the 'high-low' method of 191 to estimate the range of possible values for ϵ_T, given the range of possible values for T and $\mathcal{A}$ (assume the error in power is negligible).

Richardson-Dushman Law

You should have already checked for an approximately exponential-inverse relationship between T and I_A, edited out aberrant data (e.g., not yet plateaued), and have a range of possible values (due to systematic error in temperature) for k_2 in the expected relationship:

$$I_A = k_1\, \mathcal{A}T^2\, e^{-k_2/T} \tag{3.79}$$

We now seek a treatment incorporating the hysteresis error in I_A and the proper functional form, to obtain the best possible value for k_2. We will need to manipulate[14] the data in `rich.dat` to bring together equivalent data from the temperature upsweep and downsweep. An easy way to get the data into the `gnumeric` spreadsheet, is to type it to the screen using the UNIX command[15] `cat`:

```
cat rich.dat
```

You can then copy and paste[16] this data into `gnumeric`. Our aim to put together data with the same filament current, i.e., to combine the first line of data with the last; to combine the second line of data with the next-to-last; etc. This is easily accomplished by typing the data bottom-to-top to the screen using the UNIX command[17] `tac`:

```
tac rich.dat
```

The results can be copy and pasted next to the previous data so that the data we aim to combine is on the same line. The best estimate for T is the average of the four Ts (the two T_i should be nearly identical); the best estimate for I_A is the average of the two I_As; for δI_A use half the difference between the two values of I_A ($|I_{A1} - I_{A2}|/2$); for δT use half the difference between T_r and T_i (either pair or average the two differences). The result of this 'data reduction' is half as many data points, but with a value for δI_A based on hysteresis. (The alternative for δI_A would be the meter manufacturer's specifications, and we know they are way to small from analysis of Child's Law.)

Report the relative importance of hysteresis and calibration in temperature uncertainty determination. For $I_f = 1.2$ A record the difference in T_r due to hysteresis and the difference between T_r and T_i which is a temperature calibration uncertainty.

Copy and paste this reduced data into a new file, and fit it to Eq. 3.79. Produce a plot similar to Fig. 3.10; include lines both for your best fit and the 'book' values of k_1 and k_2. A glace at Eq. 3.69 on page 56 shows that $k_1 = A$ (Richardson's Constant) and $kk_2 = W$ (Work Function).

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[14] This processing could very easily have been done within the program itself. I've instead opted to make the program as simple as possible at a cost of additional 'by-hand' processing in a spreadsheet.

[15] from con<u>cat</u>enate—commonly this command is used to combine several files

[16] Note use of "S<u>e</u>e two two separators as one": Alt-e

[17] clever or what?

The work function is an atomic quantity, and it is usually expressed in the atomic scale unit eV[18]. Calculate the work function from your value of k_2 in joules and eV and compare to the book[19] value of 4.5 eV. The book value for A is 0.72×10^6 A/m^{2}K^2; This will serve as a fair initial guess for k_1 (required for fitting), but we need a better way to calculate A (and particularly the error in A) based on your data[20]. Since our temperatures are so uncertain, particularly at the low end, the best estimate for the Richardson constant A comes from substituting the book value of the work function and the plateau (T, I) measurement for the highest valid filament temperature into Eq. 3.69. We can then estimate the systematic uncertainty in A by using the 'high-low' method of 191 $(A^+ \rightarrow I^+, T^-, \mathcal{A}^-)$.

Report Checklist

1. Write an introductory paragraph describing the basic physics behind this experiment. For example, why did higher cathode currents produce ever higher plateaus of anode current? (This manual has many pages on these topics; your job is condense this into a few sentences and no equations.)

2. Calculations (no errors) of room temperature R_W. Measurements (4-wire ohmmeter and direct voltage/current) of R_{measured} at room temperature. Calculation of R_{support}.

3. Observations of the light intensity and color as a function of filament temperature.

4. Data files and computer program: Leave them in your UNIX account. Print out a copy of your program, the file `filament.dat`, and the data you used to fit Richardson's Law; Tape them into your lab notebook.

5. Plots similar to Figures 3.9, 3.10 (with `fit` curve Eq. 3.68 and also the Richardson function using "book" values for k_1 and k_2), 3.11 (with `fit` curve Eq. 3.67), 3.12 (with line Eq. 3.66) and 3.13 (two separate *WAPP*$^+$ plots one with T_r data and the other with T_i data). Note that Figure 3.9 is complex to produce. Use a file of `Nplot` commands and feel free to talk to me about how to go about doing this. Carefully note the use of log and inverse scales (which requires positive data—edit to achieve this!). Also include a fit report for each fit curve.

6. Experimental values (with error range) for: W (in eV), A, e/m and ϵ_T.

7. Show the missing steps leading to Equations (3.31) and (3.36). Substitute the $\rho \rightarrow 1$ approximation (Eq. 3.35) into the differential equation Equation (3.31). Show that while we do not have an exact solution to the differential equation, the singular parts (i.e., those that approach infinity as $\rho \rightarrow 1$) cancel.

8. Make a final results table, recording your final results (with proper units and sig-figs) adjacent to the corresponding 'book' values. Don't forget to record your tube's identifying letter!

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[18] Recall: 1 eV $= 1.6022 \times 10^{-19}$ J is the energy an electron gains in going through a potential difference of 1 V.

[19] Blakemore, *Solid State Physics*

[20] Recall that when systematic errors dominate computer-based errors are usually some sort of non-sense.

Comment: Classical vs. Quantum

I said above that the presence of an $\hbar$ in Richardson's constant shows that the process is governed by quantum mechanics. It is not quite so simple. The evaporation of water is not fundamentally different from that of electrons, while the former (due to the larger mass of a water molecule) is well-approximated by a classical calculation. Classical evaporation can be calculated much as quantum evaporation using the Maxwell-Boltzmann speed distribution and the number density (of water molecules) rather than the disguised version of this: Fermi energy ($E_F^{\frac{3}{2}} \propto$ number density). We can write the classical rate in terms of the quantum with no $\hbar$ visible:

$$\text{classical flux} = \frac{4}{3\sqrt{\pi}} \left[\frac{E_F}{kT}\right]^{\frac{3}{2}} \times \text{quantum flux} \tag{3.80}$$

The different temperature dependence[21] for the classical flux ($T^{\frac{1}{2}}e^{-W/kT}$ vs. $T^2 e^{-W/kT}$) cannot be detected experimentally: the Boltzmann factor overwhelms all other temperature dependencies. On the other hand, since $E_F \gg kT$, the expected classical rate is much larger than the quantum rate. This played a role in the mistaken idea that thermionic emission was due to surface contamination: the experimental rate seemed too small to be thermal evaporation. On the other hand a more fruitful interpretation of the "low" rate was that only a fraction ($\sim kT/E_F$) of the electrons were thermally active. This connected with other observations (like the "small" specific heat of the electron gas) and provided a link to the idea of a degenerate Fermi gas.

Comment: Uncertainty

Inspection of Figures 3.9-3.12 shows that something "funny" is going on. (I can't say "abnormal" or "unusual" as it is neither.) Figure 3.12 shows the unmistakable signs of "small reduced χ^2": The fitted line goes nearly dead-center through all 30 error bars, never coming even close to an edge. For error bars sized at one standard deviation (σ), you should expect total misses of the error bar about 1/3 of the time. In addition recall that each data point is really a double: the same filament current was sourced as part of a temperature up-sweep and as part of a temperature down-sweep. These repeated measurement should also frequently miss by a standard deviation, but here they are so close that the two points often look like just one. The answer to this puzzle is that the error bars are not displaying statistical ('random') error. Instead the temperature was measured two different ways (T_i and T_r), and the error bar represented the deviation between these two measurement methods. When different methods of measurement produce different answers for the same quantity, we have a textbook example of *systematic error* (in contrast to statistical error). Notice that if we had used the statistical deviation of just one measure of temperature, we would seriously underestimated the error. Furthermore since quite accurately measured electrical quantities were used to calculate the temperature (via Equation 3.62 or Equation 3.56), application of the usual error propagation methods would also have produced very small errors in T. The full range of our uncertainty in temperature is made evident only by measuring T two different ways. (This is typically how systematic errors are detected.)

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[21] Saul Dushman (Phys. Rev. **21**, 623–636 (1923)), while working at G.E., provided a general thermodynamic argument for the T^2 dependence and the universal nature of A. The resulting relationship is therefore sometimes known as the *Richardson-Dushman* relationship.

Having detected systematic errors, we should seek an explanation... In addition to the problems previously cited (particularly use of book values for filament dimensions and the problems associated with R_{support}), nonuniform filament temperature due to filament supports may be the problem. Koller (p. 89) reports the filament temperature 0.5 cm from a support is reduced by 15% (and of course the effect is greater closer to the support). Thermionic emission is reduced a similar amount 1.3 cm from a support. Thus the quantity we are seeking (a filament temperature) does not even exist, so it is hardly surprising that different measurements give different results. (It turns out the T_r is something like the average temperature; whereas T_i is something like the central temperature.) These effects have been investigated, and Koller gives the theory to correct such measurements, but such considerations are beyond the scope of this experiment.

Figure 3.9 shows the opposite problem, "large reduced χ^2": The fitted line systematically misses almost every error bar. In this case, the miss might be called "small", but the error bar is smaller still. Once again, errors were not calculated statistically (manufacturer's specifications were used), so "reduced $\chi^2 = 1$" should not really be expected. In this case, my guess is that the problem is with our simplified theory (we assumed: infinite cylinders, no random component to the electron velocity [zero electron temperature], uniform filament [temperature, voltage, emissivity, ...], perfect vacuum, no incipient current plateau). We could of course test these hypotheses by further experiments with different tubes, but such work is beyond the scope of this experiment. (Indeed I have detected V_A-sweep hysteresis; correction for this dramatically reduces reduced chi^2, but not all the way to ~ 1.)

Summary: Very large or very small reduced χ^2 suggests significant non-statistical errors, a very common —perhaps even the usual— situation. Computer generated errors are some sort of none sense in this circumstance. Presumably your theory and/or measurement methods are less than perfect. That is, of course, No Surprise. Science next requires you to guess and perhaps further investigate which imperfections are the leading source of the problems, i.e., what changes to the experiment would ameliorate the problem.

References

1. Jones & Langmuir *GE Review*, The Characteristics of Tungsten Filaments as Functions of Temperature **30** (1927) Part I pp. 310–19, Part II pp. 354–61, Part III pp. 408–12
2. Melissinos *Experiments in Modern Physics*, 1966, pp. 65–80
3. Preston & Dietz *The Art of Experimental Physics*, 1991, pp. 141–47, 152–61
4. Blakemore *Solid State Physics*, 1974, pp. 188–95
5. Koller *The Physics of Electron Tubes*, 1937, Ch. I, VI, VIII
6. G.E. *FP-400 Description and Rating* `FP400.pdf` found at `http://www.tubedata.org`

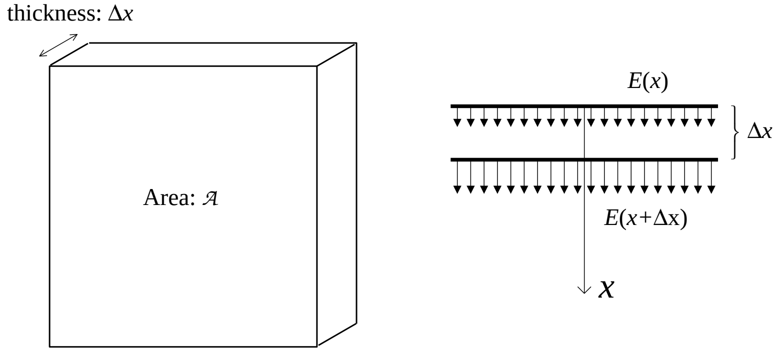

Figure 3.14: Gauss' Law is used to calculate the charge between two plates of area $\mathcal{A}$ separated by a distance Δx. Since (by assumption) the potential just depends on x, the electric field is in the x direction and is given by $E = -dV/dx$.

Appendix—Poisson's Equation

Equation 3.5 and Equation 3.28 made reference to "Poisson's Equation", which is really a topic for Physics 341, rather than part of Physics 200. In this appendix, Poisson's Equation is derived starting from two Physics 200 results: Gauss' Law: that the electric flux leaving a region depends just on the charge enclosed by that region:

$$\oint \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = Q_{\text{enclosed}}/\epsilon_0 \tag{3.81}$$

and the relationship between electric field and electric potential (voltage):

$$E_x = -\frac{dV}{dx} \tag{3.82}$$

Poisson's Equation is a differential equation equivalent to Gauss' Law. It is usually written in terms of the Laplacian (∇^2), which in turn can most easily be expressed in terms of second derivatives w.r.t. x, y, and z:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\rho/\epsilon_0 \tag{3.83}$$

where ρ is the electric charge density. We need Poisson's Equation only in cases where the electric potential depends on just one variable (x or cylindrical r), which simplifies the required proofs considerably.

As shown in Figure 3.14, if V is a function of x alone: $V(x)$, we can find the charge between two plates of area $\mathcal{A}$ using Gauss' Law:

$$Q_{\text{enclosed}} = \epsilon_0 \mathcal{A}\left(E(x+\Delta x) - E(x)\right) \approx \epsilon_0 \mathcal{A}\, \frac{dE}{dx}\, \Delta x \tag{3.84}$$

Thus the charge density between the plates is given by:

$$\rho = \frac{Q_{\text{enclosed}}}{\text{volume}} = \frac{\epsilon_0 \mathcal{A}\, \frac{dE}{dx}\, \Delta x}{\mathcal{A}\Delta x} = \epsilon_0\, \frac{dE}{dx} = -\epsilon_0\, \frac{d^2 V}{dx^2} \tag{3.85}$$

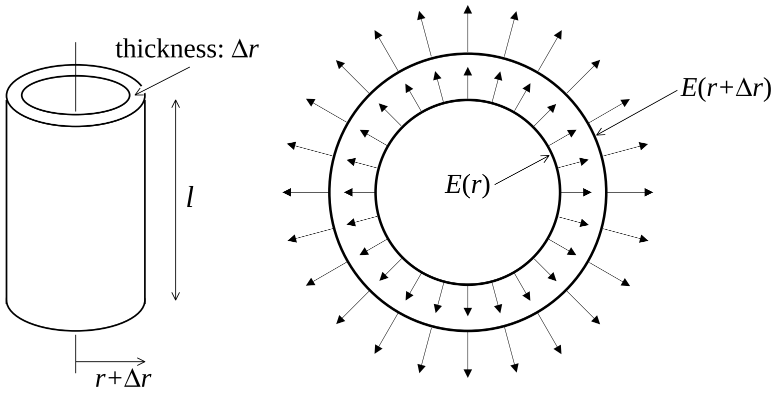

Figure 3.15: Gauss' Law is used to calculate the charge between two coaxial cylinders of length l separated by a distance Δr. Since (by assumption) the potential just depends on r, the electric field is in the r direction and is given by $E = -dV/dr$.

which provides what is needed for Equation 3.5.

As shown in Figure 3.15, if V is a function of r alone: $V(r)$, we can find the charge between two coaxial cylinders using Gauss' Law:

$$\begin{aligned} Q_{\text{enclosed}} &= \epsilon_0 l \, \{2\pi(r+\Delta r)E(r+\Delta r) - 2\pi r E(r)\} \\ &= \epsilon_0 l \, \{2\pi r(E(r+\Delta r) - E(r)) + 2\pi \Delta r E(r+\Delta r)\} \\ &\approx \epsilon_0 l \left\{ 2\pi r \, \frac{dE}{dr} + 2\pi E(r) \right\} \Delta r \end{aligned} \tag{3.86}$$

Thus the charge density between the cylinders is given by:

$$\rho = \frac{Q_{\text{enclosed}}}{\text{volume}} = \frac{\epsilon_0 l \, \{2\pi r \, dE/dr + 2\pi E(r)\} \Delta r}{2\pi r l \Delta r} = \epsilon_0 \left\{ \frac{dE}{dr} + \frac{1}{r} \, E \right\} \tag{3.87}$$

$$= -\,\epsilon_0 \left\{ \frac{d^2V}{dr^2} + \frac{1}{r} \, \frac{dV}{dr} \right\} \tag{3.88}$$

which provides what is needed for Equation 3.28.